Normal Forms, Connections and Chains on Nondegenerate CR Manifolds

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摘要

在這篇論文中，我們研習非退化柯西黎曼流形(簡稱CR流形)的不變量。解答局部等價問題我們可考慮兩個方向。於外在方向，我們定義於空間裡超實曲面定義函數的正規形式。於內在方向，我們在CR流形的一主叢上構造一個連絡。我們會詳細研習這兩個解答和它們的比較。

我們在兩個解答中各發現了稱爲鏈的不變曲線。鏈的引入非常自然，但在上述兩個方向中的定義手法不同。最初希望它們與黎曼幾何中的測地線相似，但結果發現它們比較複雜。

我們會先討論非退化CR流形的典型例子超二次曲面。之後會討論上述兩個局部等價問題的解答及鏈的特性。

Abstract

In this thesis, we study invariants of nondegenerate CR manifolds. There are two approaches to solve the local equivalence problem of nondegenerate CR manifolds. In the extrinsic approach, one defines normal forms of the defining function of real hypersurfaces in $\mathbb{C}^{n+1}$. In the intrinsic approach, one constructs a connection on a principal bundle over the CR manifolds. We study in detail both approaches and their comparision.

Both approaches reveal a set of invariant curves called chains. These chains occur naturally but in very different manners in the two approaches. They are at first compared to geodesics in Riemannian geometry but their behaviour turns out more complicated.

The thesis first studies the hyperquadrics, which are the model space of non-degenerate CR manifolds. Then the two approaches are presented and the chains are investigated.

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Contents

Introduction

In this thesis, we study the equivalence problem of non-degenerate real analytic hypersurfaces in $\mathbb{C}^{n+1}$. In 1907, Poincaré showed that two real hypersurfaces in $\mathbb{C}^2$ are in general biholomorphically inequivalent. Precisely, there usually is no local biholomorphism between two given real analytic submanifolds of real dimension three in $\mathbb{C}^2$. Then he raised the question of finding the invariants distinguishing two real hypersurfaces.

It turns out that the solutions to the problem may be given in two quite different approaches, one of which was developed by E. Cartan in 1932 [Ca]. His solution was given as a complete set of invariant one forms on a principal bundle Y over the real hypersurface in $\mathbb{C}^2$. In a paper of Chern and Moser in 1974 [CM], the solution was generalized to higher dimensional cases. In the same paper, a different approach was provided, in which the biholomorphism invariants are the coefficients of the Taylor series of the defining function of the real hypersurface. Both approaches unveil respective families of invariant curves, called chains, which turn out to be the same. Chains were initially compared to geodesics in Riemannian geometry. Fefferman later showed that chains may have pathological behaviour.

We begin chapter 1 of the thesis by giving Poincaré's arguments on the non-equivalence of real hyperfuraces in $\mathbb{C}^2$. Then the definition and some basic properties of CR manifolds will be presented as generalization of real hypersurfaces in $\mathbb{C}^{n+1}$. Since the real hyperquadrics, its automorphism group $SU(p+1, q+1)$ and the isotropy group H of the origin play a fundamental role in the solution of equivalence problem, they are discussed in details.

In chapter 2 we show how a normal form is obtained by approximating locally a nondegenerate real hypersurface to a real hyperquadric. The tangential direction transversal to the complex tangent space is treated by assigning double weight. Following [CM], we study first the formal theory and then the geometric theory.

In chapter 3, we study the Chern-Moser solution of the equivalence problem. A complete set of invariant forms on a principal bundle Y is determined and these are interpreted as a connection with the group $SU(p+1, q+1)$.

In the last chapter, we first show that the two sets of invariant curves coincide. Then some basic properties and pathological behaviour will be presented.

Chapter 1

Introduction to CR manifolds

1.1 Non-equivalence of real analytic hypersurfaces in $\mathbb{C}^2$

In $\mathbb{C}$, any two real analytic curves are locally biholomorphic. It is natural to ask whether the result can be generalized to higher dimension cases. In fact, Poincaré showed that analogous result does not hold even in $\mathbb{C}^2$. He gave two local arguments:

<u>Argument 1</u>

Let

$$\begin{aligned} s &= \{(z,w) = (x+iy, u+iv) \in \mathbb{C}^2 \mid v = \phi(x,y,u)\} \\ S &= \{(Z,W) = (X+iY, U+iV) \in \mathbb{C}^2 \mid V = \Phi(X,Y,U)\} \end{aligned}$$

be two real analytic hypersurfaces in $\mathbb{C}^2$ and let $f = (f_1, f_2)$ be a local biholomorphism which takes s to S.

Since $(f_1(x+iy, u+i\phi(x,y,u)), f_2(x+iy, u+i\phi(x,y,u))) \in S$, one has

$$\begin{aligned} F_1(x,y,u) &=: X(x,y,u) + iY(x,y,u) = f_1(x+iy, u+i\phi(x,y,u)) \\ F_2(x,y,u) &=: U(x,y,u) + i\Phi(X,Y,U) = f_2(x+iy, u+i\phi(x,y,u)) \end{aligned}$$

By the chain rule, for j=1, 2

$$
\begin{aligned}
\frac{\partial F_j}{\partial x} &= \frac{\partial f_j}{\partial z} + \frac{\partial f_j}{\partial w} i\phi_x \\
\frac{\partial F_j}{\partial y} &= \frac{\partial f_j}{\partial z} i + \frac{\partial f_j}{\partial w} i\phi_y \\
\frac{\partial F_j}{\partial u} &= \frac{\partial f_j}{\partial w}(1 + i\phi_u)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \frac{\partial F_j}{\partial \bar{z}} &= \frac{\partial f_j}{\partial w} i\phi_{\bar{z}} \\
\frac{\partial F_j}{\partial u} &= \frac{\partial f_j}{\partial w}(1 + \phi_u)
\end{aligned}
$$

$$
\Rightarrow LF_j =: ((1 + i\phi_u)\frac{\partial}{\partial \bar{z}} - i\phi_{\bar{z}}\frac{\partial}{\partial u})F_j = 0, \quad j = 1, 2 \tag{1.1}
$$

For s and S to be locally biholomorphic we need to find 3 real functions X, Y, U of the variables (x,y,u) satisfying (1.1). We have 4 real equations from (1.1) for 3 real unknowns. Thus solutions usually do not exist.

Argument 2

The second is a counting argument which uses the fact that the Nth degree Taylor polynomial of a function of k variables contains $\begin{pmatrix} N+k \\ k \end{pmatrix}$ unlike terms. Let

$$
M_0 = \{(z, w) = (x + iy, u + iv) \in \mathbb{C}^2 \mid v = F_0(x, y, u), 0 = F_0(0, 0, 0)\}
$$

and $\mathcal{M}$ be the set of nearby surfaces of the same form

$$
\mathcal{M} = \{(z, w) = (x + iy, u + iv) \in \mathbb{C}^2 \mid v = F(x, y, u)\}
$$

Since the Taylor expansion coefficient of order $\leq$N at the origin of F contains p = $\begin{pmatrix} N+3 \\ 3 \end{pmatrix}$ terms. There is a map form $\mathcal{M}$ into $\mathbb{R}^p$. The map covers a neighborhood of the origin. On the other hand, let $\mathcal{S}$ be the set of local biholomorphims of $\mathbb{C}^2$ defined in a neighborhood of the origin and taking M_0 to an element in $\mathcal{M}$. Then there is a map of $\mathcal{S}$ into $\mathbb{R}^p$. This map factors through the map of $\mathcal{S}$ into the space of N-jets of pairs of holomorphic functions, namely $\mathbb{C}^q \times \mathbb{C}^q \cong \mathbb{R}^{4q}, q = \begin{pmatrix} N+2 \\ 2 \end{pmatrix}$. If M_0 were equivalent to every other hypersurface of the same form there would be a continuous map form $\mathbb{R}^q$ to $\mathbb{R}^p$ which covers some neighborhood of the origin. But this is impossible whenever $N > 10$ because in this case $p > 4q$.

Since not all hypersurfaces are locally equivalent it is natural to seek invariants which allow us to distinguish one from another. The problem of finding these invariants in $\mathbb{C}^2$ was solved by Cartan and generalized to higher dimensions of Chern and Moser. We will study their results in the following chapters.

1.2 CR manifold and Levi form

In this section, we will give the definitions and some basic properties of CR manifolds. To simplify the notations, in what follows small Greek indices run from 1 to n and A, B run from 0 to n+1. We also fix the reference coordinates (z^α, w) for $\mathbb{C}^{n+1}$, $z = (z^\alpha) \in \mathbb{C}^n$, $w = u + iv \in \mathbb{C}$.

Definition 1.2.1 *Let M be an abstract C^∞ manifold with $\dim_{\mathbb{R}} M = 2n+1$. Let $\mathbb{L}$ be a subbundle of the complexified tangent bundle $T^{\mathbb{C}}M$ with $\dim_{\mathbb{C}} \mathbb{L} = n$. $(M, \mathbb{L})$ is said to be a CR manifold if $\mathbb{L} \cap \bar{\mathbb{L}} = \{0\}$ and $\mathbb{L}$ is involutive, ie $[L_1, L_2] \in \mathbb{L}$ whenever $L_1, L_2 \in \mathbb{L}$. With $\mathbb{L}$ understood we call M a CR manifold.*

Remarks

(i) Almost everything we mention will be local in nature. So often we refer to $(M, \mathbb{L})$ as a CR structure rather than a CR manifold to emphasize that our study is of local properties rather than of global objects in the manifold.

(ii) It is worth to note that in some definitions, the conditions that $\dim_{\mathbb{R}} M = 2n+1$ and $\dim_{\mathbb{C}} \mathbb{L} = n$ are dropped. In such cases, we call $\dim_{\mathbb{C}}\{T^{\mathbb{C}}M/\mathbb{L} \oplus \bar{\mathbb{L}}\}$ the CR codimension of $(M, \mathbb{L})$. Any complex manifold is a CR manifold of codimension 0 in this sense. Definition 1.2.1 gives CR manifolds of codimension 1.

(iii) If $(M, \mathbb{L})$ defines a CR structure, so does $(M, \bar{\mathbb{L}})$ which is called the conjugate CR structure with respect to the original one.

(iv) There is a complex structure map J defined on the real subbundle H of TM which generates $\mathbb{L}$ so that $\mathbb{L}$ and $\bar{\mathbb{L}}$ are the -i and +i eigenspaces of the extension of J to $\mathbb{L} \oplus \bar{\mathbb{L}}$ respectively.
Precisely, we choose some basis $\{\mathbb{L}_\alpha\}$ for $\mathbb{L}$ and note that $\mathbb{L} \cap \bar{\mathbb{L}} = \{0\}$ implies $\{Re\mathbb{L}_\alpha, Im\mathbb{L}_\alpha\}$ is linearly independent over $\mathbb{R}$. We set $H = span_{\mathbb{R}}\{Re\mathbb{L}_\alpha, Im\mathbb{L}_\alpha\}$.

H is a real 2n dimensional subbundle of TM which does not depend on the choice of basis. Define J by

$$\begin{aligned} J(ReL_\alpha) &= ImL_\alpha \\ J(ImL_\alpha) &= -ReL_\alpha \end{aligned}$$

Then the extension of J to $\mathbb{C} \otimes H$ maps L_α to $-iL_\alpha$ and $\bar{L}_\alpha$ to $i\bar{L}_\alpha$, which shows $\mathbb{L}$ and $\bar{\mathbb{L}}$ are the $-i$ and $+i$ eigenspaces of J as claimed. Moreover, for $X, Y \in H$, $X + iJX, Y + iJY \in \mathbb{L}$. Then

$$\begin{aligned} & [X + iJX, Y + iJY] \\ = \;& [X, Y] - [JX, JY] + i([X, JY] + [JX, Y]) \end{aligned}$$

Therefore the integrability condition $[\mathbb{L}, \mathbb{L}] \subset \mathbb{L}$ is equivalent to

$$[X, JY] + [JX, Y] \in H$$
$$[JX, JY] - [X, Y] = J([X, JY] + [JX, Y]) \in H$$

(v) The class of all real hypersurfaces in $\mathbb{C}^{n+1}$ are CR manifolds. The subbundle $\mathbb{L}$ is defined by $H_{0,1}(\mathbb{C}^{n+1}) \cap T^{\mathbb{C}}M$, where

$$H_{0,1}(\mathbb{C}^{n+1}) =: span_{\mathbb{C}} \frac{\partial}{\partial z^{\bar{\alpha}}}$$

For a real hypersurface $M = \{(z, w) \mid v = F(z, \bar{z}, u)\}$, we claim that a basis for $\mathbb{L}$ is given by

$$L_\alpha = (1 + iF_u)\frac{\partial}{\partial z^{\bar{\alpha}}} - iF_{z^{\bar{\alpha}}}\frac{\partial}{\partial u}$$

To see this, we note that the defining function for M is given by

$$r = \frac{w - \bar{w}}{2i} - F(z, \bar{z}, \frac{w + \bar{w}}{2})$$

and we want to find L_α such that $L_\alpha r = 0$. So we can take

$$\begin{aligned} L_\alpha &= r_{\bar{w}}\frac{\partial}{\partial z^{\bar{\alpha}}} - r_{z^{\bar{\alpha}}}\frac{\partial}{\partial \bar{w}} \\ &= (-\frac{1}{2i} - \frac{1}{2}F_u)\frac{\partial}{\partial z^{\bar{\alpha}}} + F_{z^{\bar{\alpha}}}\frac{\partial}{\partial \bar{w}} \end{aligned}$$

Let f be a function on M, in order to compute $L_\alpha f$ we extend f to be indpendent of v so that $f_{\bar{w}} = \frac{1}{2} f_u$. Multiplying the factor of $-2i$, one has

$$L_\alpha = (1 + iF_u) \frac{\partial}{\partial z^{\bar{\alpha}}} - iF_{z^{\bar{\alpha}}} \frac{\partial}{\partial u}$$

One of the defining properties of an abstract CR manifold $(M, \mathbb{L})$ is that $\mathbb{L}$ is involutive. The subbundle $\mathbb{L} \oplus \bar{\mathbb{L}} \subset T^{\mathbb{C}}M$ is not necessarily involutive. Let $(M, \mathbb{L})$ be a CR structure. Let $\{L_\alpha\}$ be a basis for $\mathbb{L}$ and U be a real vector transversal to $\mathbb{L} \oplus \bar{\mathbb{L}}$. Let

$$[L_\alpha, L_{\bar{\beta}}] = i g_{\alpha\bar{\beta}} U \quad mod\, \mathbb{L} \oplus \bar{\mathbb{L}}$$

defines the matrix g. This matrix is often referred to as the Levi form.

Remarks

(i) g is hermitian since

$$\begin{aligned} \overline{i g_{\alpha\bar{\beta}} U} &= \overline{[L_\alpha, L_{\bar{\beta}}]} \\ &= -[L_\beta, L_{\bar{\alpha}}] \\ &= -i g_{\beta\bar{\alpha}} U \quad mod\, \mathbb{L} \oplus \bar{\mathbb{L}} \end{aligned}$$

Thus $g_{\alpha\bar{\beta}} = \overline{g_{\beta\bar{\alpha}}}$.

(ii) The number of positive, negative and zero eigenvalues (p, q, r) are independent of the choice of basis of $\mathbb{L}$ and U. To see this, assume $\{\tilde{L}_\rho, \tilde{U}\}$ be another choice with

$$\begin{aligned} \tilde{L}_\rho &= A_\rho^\sigma L_\sigma \\ \tilde{U} &= bU \quad mod\, \mathbb{L} \oplus \bar{\mathbb{L}} \end{aligned}$$

where $A = (A_\rho^\sigma)$ is invertible and b is a non-zero real number. We have

$$\begin{aligned} i \tilde{g}_{\rho\bar{\sigma}} \tilde{U} &= [\tilde{L}_\rho, \tilde{L}_{\bar{\sigma}}] \\ &= [A_\rho^\alpha L_\alpha, A_{\bar{\sigma}}^{\bar{\beta}} L_{\bar{\beta}}] \\ &= i b^{-1} A_\rho^\alpha g_{\alpha\bar{\beta}} A_{\bar{\sigma}}^{\bar{\beta}} \tilde{U} \quad mod\, \mathbb{L} \oplus \bar{\mathbb{L}} \end{aligned}$$

so $\tilde{g} = b^{-1}AgA^*$ has the same signature as g.

(iii) A CR manifold $(M, \mathbb{L})$ is Levi-flat if the Levi form of M vanishes at each point in M. This implies $\mathbb{L} \oplus \bar{\mathbb{L}}$ is involutive.

Definition 1.2.2 *A CR manifold M is nondegenerate at $p \in M$ if the Levi form g_p is non-singular. M is called strictly pseudoconvex at p if g_p is either positive or negative definite. M is called a nondegenerate (respectively strictly pseudoconvex) CR manifold if M is nondegenerate (respectively strictly pseudoconvex) at all $p \in M$.*

The definitions of CR manifolds and their Levi forms can be formulated in terms of differential forms.

Let $(M, \mathbb{L})$ be a CR structure. We choose local one-forms $(\theta, \theta^\alpha, \theta^{\bar{\alpha}}) \in T^{*\mathbb{C}}M$ such that

(i) θ is real and $\theta(X) = 0$ for all $X \in \mathbb{L} \oplus \bar{\mathbb{L}}$;

(ii) $span(\theta, \theta^\alpha) = L^\perp$, ie. $\theta \wedge \theta^1 \wedge ... \wedge \theta^n \wedge \theta^{\bar{1}} \wedge ... \wedge \theta^{\bar{n}} \neq 0$;

(iii) $d\theta \equiv d\theta^\alpha \equiv 0 \quad mod\ \theta, \theta^\alpha$;

Conversely, given $(\theta, \theta^\alpha, \theta^{\bar{\alpha}})$ we take $\mathbb{L} = \{\theta, \theta^\alpha\}^\perp$. Since $d\theta \equiv d\theta^\alpha \equiv 0 \quad mod\ \theta, \theta^\alpha$, we have

$$\begin{aligned} d\theta &= ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + \theta \wedge (\eta_\alpha \theta^\alpha + \eta_{\bar{\alpha}}\theta^{\bar{\alpha}}) \\ &\equiv ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} \quad mod\ \theta \end{aligned}$$

Proposition 1.2.3 *Let M be a CR manifold with CR strucutre $(\theta, \theta^\alpha, \theta^{\bar{\alpha}})$. Write $d\theta \equiv ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} \quad mod\ \theta$. Then $(h_{\alpha\bar{\beta}})$ defines a Levi form for M.*

Proof: Let $\mathbb{L} = (\theta, \theta^\alpha)^\perp$. We choose $(U, L_\alpha, L_{\bar{\beta}})$ such that:

(i) $\{L_\alpha\}$ is a basis for $\mathbb{L}$

(ii) $[L_\alpha, L_{\bar{\beta}}] = ig_{\alpha\bar{\beta}}U \quad mod\ \mathbb{L} \oplus \bar{\mathbb{L}}$

(iii) $\theta(U) = 1, \quad \theta^\alpha(L_{\bar{\beta}}) = \theta^{\bar{\alpha}}(L_\beta) = \delta^\alpha_\beta$

For $L \in \mathbb{L}$, we define a by $[L, \bar{L}] = iaU \quad mod\, \mathbb{L} \oplus \bar{\mathbb{L}}$. Then

$$\begin{aligned} ia &= \theta[L, \bar{L}] \\ &= -d\theta(L, \bar{L}) + L\theta(\bar{L}) - \bar{L}\theta(L) \\ &= ih_{\alpha\bar{\beta}}\theta^{\beta}(\bar{L})\theta^{\bar{\alpha}}(L) \end{aligned}$$

On the other hand, since $\theta^{\alpha}(L_{\bar{\beta}}) = \theta^{\bar{\alpha}}(L_{\beta}) = \delta^{\alpha}_{\beta}$, we have $L = \theta^{\bar{\alpha}}(L)L_{\alpha}$. Then

$$\begin{aligned} iaU &= [L, \bar{L}] \\ &= [\theta^{\bar{\alpha}}(L)L_{\alpha}, \theta^{\beta}(\bar{L})L_{\bar{\beta}}] \\ &= \theta^{\bar{\alpha}}(L)\theta^{\beta}(\bar{L})[L_{\alpha}, L_{\bar{\beta}}] \\ &= i\theta^{\bar{\alpha}}(L)\theta^{\beta}(\bar{L})g_{\alpha\bar{\beta}}U \end{aligned}$$

Therefore we have $(h_{\alpha\bar{\beta}}) = (g_{\alpha\bar{\beta}})$. $\square$

Remarks

(i) The condition $d\theta \equiv d\theta^{\alpha} \equiv 0 \quad mod\ \theta, \theta^{\alpha}$ corresponds to the integrability condition of $\mathbb{L}$.

(ii) For $M = \{(z, w) \mid r(z, \bar{z}, w, \bar{w}) = 0\}$, we may take $\theta = i\partial r$. Since $0 = dr(X) = \partial r(X) + \bar{\partial} r(X)$ for all $X \in TM$, we have $i\partial r(X)$ is real. Besides, for $Y \in H$, $Y + iJY \in \mathbb{L} \subset H_{0,1}(\mathbb{C}^{n+1})$, so $\partial r(Y + iJY) = 0$ and thus $\partial r Y = 0$.

(iii) For $M = \{(z, w) \mid v = F(z, \bar{z}, u) = 0\}$, we can take

$$\begin{aligned} \theta &= -\frac{1}{2}(1 + {F_u}^2)du + \frac{1}{2}(iF_{\bar{\alpha}} - F_u F_{\bar{\alpha}})dz^{\alpha} + \frac{1}{2}(-iF_{z^{\bar{\alpha}}} - F_u F_{\bar{\alpha}})dz^{\bar{\alpha}} \\ \theta^{\alpha} &= dz^{\alpha} \end{aligned}$$

By the above remark, we have

$$\begin{aligned}
\theta &= i\partial r \\
&= i\partial(-\frac{-(w-\bar{w})}{2i} + F(z,\bar{z},u)) \\
&= iF_\alpha dz^\alpha + (\frac{i}{2}F_u - \frac{1}{2})dw
\end{aligned}$$

$$\begin{aligned}
dw &= du + idF \\
&= (1+iF_u)du + iF_{z^\alpha}dz^\alpha + iF_{z^{\bar{\alpha}}}dz^{\bar{\alpha}}
\end{aligned}$$

$$\Rightarrow \theta = -\tfrac{1}{2}(1+{F_u}^2)du + \tfrac{1}{2}(iF_{\bar{\alpha}} - F_uF_{\bar{\alpha}})dz^\alpha + \tfrac{1}{2}(-iF_{z^{\bar{\alpha}}} - F_uF_{\bar{\alpha}})dz^{\bar{\alpha}}$$

On the other hand, the restrictions to M of the forms dz^α annihilate $\mathbb{L}$ and satisfy

$$\theta \wedge dz^1 \wedge ... \wedge dz^n \wedge dz^{\bar{1}} \wedge ... \wedge dz^{\bar{n}} \neq 0$$

$$d\theta \equiv dz^\alpha \equiv 0 \quad mod\ \theta, dz^\alpha$$

Therefore $(\theta, \theta^\alpha, \theta^{\bar{\alpha}})$ defines a CR structure as claimed.

Proposition 1.2.4 *For $M = \{(z,w) \mid v = F(z,\bar{z},u)\}$, $g_{\alpha\bar{\beta}} = \left(\frac{\partial^2 F}{\partial z^\alpha \partial z^{\bar{\beta}}}\right)$ defines a Levi form for M.*

Proof: We can take $(\theta, \theta^\alpha, \theta^{\bar{\alpha}})$ as follows

$$\theta = -iF_{z^{\bar{\alpha}}}dz^\alpha - (\frac{i}{2}F_u - \frac{1}{2})dw$$

$$\theta^\alpha = dz^\alpha$$

defines a CR structure on M. Then

$$\begin{aligned}
d\theta(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^{\bar{\beta}}}) &= \frac{\partial}{\partial z^\alpha}\theta\left(\frac{\partial}{\partial z^{\bar{\beta}}}\right) - \frac{\partial}{\partial z^{\bar{\beta}}}\theta\left(\frac{\partial}{\partial z^\alpha}\right) - \theta[\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^{\bar{\beta}}}] \\
&\equiv \frac{\partial}{\partial z^{\bar{\beta}}} iF_{z^\alpha} \\
&\equiv iF_{z^\alpha z^{\bar{\beta}}} \quad mod\ \theta
\end{aligned}$$

Hence $d\theta \equiv iF_{z^\alpha z^{\bar{\beta}}}dz^\alpha dz^{\bar{\beta}} \quad mod\ \theta$ and $(F_{z^\alpha z^{\bar{\beta}}})$ defines a Levi form for M. $\square$

We usually will be concerned with local questions. Therefore we write $(M, \mathbb{L}, p)$ to denote the CR structure on M in the neighborhood of one of its points p. The notation (M,p) refers to a neighborhood of p of the manifold M.

Definition 1.2.5 *A complex-valued function f defined on a neighborhood of $p \in M$ is a CR function on $(M, \mathbb{L}, p)$ if there exists a neighborhood U_p of p such that $L(f) = 0$ for all sections $L : U_p \to \mathbb{L}$.*

Remarks

(i) f is a CR function with respect to the CR structure $(\theta,\ \theta^\alpha,\ \theta^{\bar{\alpha}})$ if and only if $df \in span(\theta, \theta^\alpha)$.

(ii) If f is a CR function, then $f_*(L) = L(f)\frac{\partial}{\partial z} + L(\bar{f})\frac{\partial}{\partial \bar{z}} = L(\bar{f})\frac{\partial}{\partial \bar{z}}$ for all $L \in \mathbb{L}$.

The notation $F : (M_1, p) \to (M_2, q)$ means that F is a map of an open set in one manifold M_1 into another manifold M_2 with $F(p) = q$

Definition 1.2.6 *A map $F : (M_1, p) \to (M_2, q)$ is a CR map with respect to the CR structures $(M_1, \mathbb{L}_1, p)$ and $(M_2, \mathbb{L}_2, q)$ if there exists a neighborhood U_p such that $F_*\mathbb{L}_1 \subset \mathbb{L}_2$ at all points of U_p.*

Remark

If F is also a diffeomorphism then it is called a CR diffeomorphism and that (M_1, p) and (M_2, q) are CR diffeomorphic.

1.3 The real hyperquadrics

The simplest domain in $\mathbb{C}^{n+1}$ is the unit ball

$$z^1 z^{\bar{1}} + ... + z^n z^{\bar{n}} + w\bar{w} < 1$$

The boundary sphere is highly symmetric. It turns out that for the study of CR manifolds, the real hyperquadric is more suitable to better reflect the difficult tangential directions transversal to the complex tangent space. Thus, under the biholomorphism

$$(z^\alpha, w) \mapsto \left(\frac{iz^\alpha}{1-w}, \frac{i(1+w)}{1-w}\right)$$

one transforms the ball to the domain

$$Imw > z^1 z^{\bar{1}} + ... + z^n z^{\bar{n}}$$

and considers the real hyperquadric

$$Imw = z^1 z^{\bar{1}} + ... + z^n z^{\bar{n}}$$

In the following we consider a slightly more general case

$$\{(z, w) \in \mathbb{C}^{n+1} \mid v = h_{\alpha\bar{\beta}} z^\alpha z^{\bar{\beta}}\}$$

where $h_{\alpha\bar{\beta}}$ are constants satisfying

$$\overline{h_{\alpha\bar{\beta}}} = h_{\beta\bar{\alpha}} \quad det(h_{\alpha\bar{\beta}}) \neq 0$$

We assume $(h_{\alpha\bar{\beta}})$ have p positive and q negative eigenvalues, $p + q = n$. To introduce a group which acts on the real hyperquadric, one imbeds $\mathbb{C}^{n+1}$ in the complex projective space $\mathbb{CP}^{n+1}$ under the transformation

$$(z = \frac{\zeta^\alpha}{\zeta^0}, w = \frac{\zeta^{n+1}}{\zeta^0}) \mapsto [\zeta^0, \zeta^1, ..., \zeta^{n+1}]$$

Then Q becomes

$$\tilde{Q} =: \{[\zeta^0, \zeta^1, ..., \zeta^{n+1}] \in \mathbb{CP}^{n+1} \mid h_{\alpha\bar\beta}\zeta^\alpha\zeta^{\bar\beta} + \frac{i}{2}(\zeta^{\bar 0}\zeta^{n+1} - \zeta^0\overline{\zeta^{n+1}}) = 0\}$$

$$= \left\{ [\zeta^0, \zeta^1, ..., \zeta^{n+1}] \in \mathbb{CP}^{n+1} : (\zeta^0, \zeta^1, ..., \zeta^{n+1}) \begin{pmatrix} 0 & 0 & -i/2 \\ 0 & h_{\alpha\bar\beta} & 0 \\ i/2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta^{\bar 0} \\ \vdots \\ \zeta^{\overline{n+1}} \end{pmatrix} = 0 \right\}$$

Thus we compactify Q by adding the following points at infinity

$$\tilde{Q} \setminus Q = \left\{ [\zeta^0, \zeta^1, ..., \zeta^{n+1}] \in \mathbb{CP}^{n+1} | h_{\alpha\bar\beta}\zeta^\alpha\zeta^{\bar\beta} = 0 \right\}$$

Further, there is a natural hermitian scalar product on $\mathbb{C}^{n+2}$ accociated with the equation of $\tilde{Q}$ in homogeneous coordinates. Namely, for two vectors $Z = (\zeta^0, \zeta^1, ..., \zeta^{n+1}), Z' = (\zeta'^0, \zeta'^1, ..., \zeta'^{n+1})$

$$(Z, Z') = h_{\alpha\bar\beta}\zeta^\alpha\zeta'^{\bar\beta} + \frac{i}{2}(\zeta'^{\bar 0}\zeta^{n+1} - \zeta^0\zeta'^{\overline{n+1}}) = 0$$

In what follows we will denote $\mathcal{H}$

$$\mathcal{H} = (h_{A\bar B}) = \begin{pmatrix} 0 & 0 & -i/2 \\ 0 & h_{\alpha\bar\beta} & 0 \\ i/2 & 0 & 0 \end{pmatrix}, 0 \le A, B \le n+1 \tag{1.2}$$

Definition 1.3.1 *Let $SU(p+1, q+1)$ be the group of unimodular linear homogeneous transformation on ζ^A which leave the form (Z, Z) invariant. More explicitly, $SU(p+1, q+1)$ may be given by*

$$\{A \in SL(n+1, \mathbb{C}) \mid A\mathcal{H}\bar{A}^T = \mathcal{H}\}$$

Note that A induces the identity map on $\mathbb{CP}^{n+1}$ if and only if $A = \varepsilon I$, $\varepsilon^{n+1} = I$.

Remarks

(i) $SU(p+1, q+1)$ is a group of automorphisms on Q. To see this, let $\zeta \in \tilde{Q}$ and $A \in SU(p+1, q+1)$. Then

$$\begin{aligned} (\zeta A)\mathcal{H}\overline{(\zeta A)^T} &= \zeta A\mathcal{H}\bar{A}^T\bar\zeta^T \\ &= \zeta\mathcal{H}\bar\zeta^T \\ &= 0 \end{aligned}$$

(ii) If K is the normal subgroup of $SU(p+1,q+1)/K$ given by $A = \varepsilon I$, $\varepsilon^{n+1} = I$, then $SU(p+1,q+1)/K$ acts on Q effectively.

Definition 1.3.2 *An ordered set of n+2 vectors $Z_0, Z_1, ..., Z_{n+1}$ in $\mathbb{C}^{n+2}$ is said to be a Q-frame if $(Z_A, Z_B) = (h_{A\bar{B}})$, $0 \le A, B \le n+1$ and $det(Z_0, Z_1, ..., Z_{n+1}) = 1$.*

Remarks

(i) $Z_0, Z_1, ..., Z_{n+1}$ is a Q-frame if and only if the matrix $(Z_0, Z_1, ..., Z_{n+1}) \in SU(p+1,q+1)$. Note that if $Z_0, Z_1, ..., Z_{n+1}$ is a Q-frame, then $Z_0, Z_{n+1} \in \tilde{Q}$.

(ii) The group $SU(p+1,q+1)$ can be identified with the space of all Q-frames, since, an element of $SU(p+1,q+1)$ may be considered as a transformation which maps a Q-frame into another. There exists exactly one transformation taking a Q-frame into another. By fixing one Q-frame as reference, the group $SU(p+1,q+1)$ is identified with the space of all Q-frames.

Let Z_A, Z_A^* be two Q-frames and let

$$Z_A^* = a_A^B Z_B$$

The linear homogeneous transformation on $\mathbb{C}^{n+1}$ which maps the frame Z_A to the frame Z_A^* maps the vector $\zeta^A Z_A$ to

$$\zeta^A Z_A^* = \zeta^A a_A^B Z_B$$

If one denotes the latter vector by $\zeta^{*B} Z_B$, one has

$$\zeta^{*B} = a_A^B \zeta^A \tag{1.3}$$

which is the most general transformation of $SU(p+1,q+1)$ when Z_A^* runs over all Q-frames.

To determine the isotropy subgroup H of $SU(p+1,q+1)$ which is the largest subgroup leaving a point Z_0 of $\tilde{Q}$ fixed, it suffices to find the most general change of Q-frames leaving the point fixed, which is

$$
\begin{aligned}
Z_0^* &= tZ_0 \\
Z_\alpha^* &= t_\alpha Z_0 + t_\alpha^\beta Z_\beta \\
Z_{n+1}^* &= \tau Z_0 + \tau^\beta Z_\beta + \bar{t}^{-1} Z_{n+1}
\end{aligned}
\tag{1.4}
$$

where

$$
\begin{aligned}
& t_\alpha = -2itt_\alpha^\rho \tau^{\bar\sigma} h_{\rho\bar\sigma} = -2itt_\alpha^\rho \tau_\rho \\
& t\bar{t}^{-1} det(t_\alpha^\beta) = 1 \\
& t_\alpha^\rho \bar{t}_\beta^\sigma h_{\rho\bar\sigma} = h_{\alpha\bar\beta} \\
& h_{\rho\bar\sigma}\tau^\rho\tau^{\bar\sigma} + \tfrac{i}{2}(\bar\tau\bar{t}^{-1} - \tau t^{-1}) = 0
\end{aligned}
\tag{1.5}
$$

H is therefore the group of all matrices

$$
\begin{pmatrix}
t & 0 & 0 \\
t_\alpha & t_\alpha^\beta & 0 \\
\tau & \tau^\beta & \bar{t}^{-1}
\end{pmatrix}
\tag{1.6}
$$

with the conditions (1.5) satisfied. Its dimension is $n^2 + 2n + 2$. By (1.3) the correpsonding coordinate change is given by:

$$
\begin{aligned}
\zeta^{*0} &= t\zeta^0 + t_\alpha\zeta^\alpha + \tau\zeta^{n+1} \\
\zeta^{*\beta} &= t_\alpha^\beta\zeta^\alpha + \tau^\beta\zeta^{n+1} \\
\zeta^{*n+1} &= \bar{t}^{-1}\zeta^{n+1}
\end{aligned}
$$

In terms of non-homogeneous coordinates:

$$
\begin{aligned}
z^{*\beta} &= (t_\alpha^\beta z^\alpha + \tau^\beta w)t^{-1}\delta^{-1} \\
w^* &= |t|^{-2} w\delta^{-1}
\end{aligned}
$$

where $\delta = 1 + t^{-1}t_\alpha\zeta^\alpha + t^{-1}\tau\omega$. By writing $C_\alpha^\beta = t^{-1}t_\alpha^\beta$, $C_\alpha^\beta a^\alpha = t^{-1}\tau^\beta$, $\rho = |t|^{-2}$, one obtains the transformations of the isotropy group H in non-homogeneous coordinates:

$$
\begin{aligned}
z^{*\beta} &= C_\alpha^\beta(z^\alpha + a^\alpha w)\delta^{-1} \\
w^* &= \rho w\delta^{-1}
\end{aligned}
\tag{1.7}
$$

where

$$C_\alpha^\lambda C_{\bar\beta}^{\bar\sigma} h_{\lambda\bar\sigma} = \rho h_{\alpha\bar\beta}$$
$$t^{-1}t_\alpha = -2ia_\alpha = -2ih_{\alpha\bar\beta}a^{\bar\beta}$$
$$Im(t^{-1}\tau) = -h_{\alpha\bar\beta}a^\alpha a^{\bar\beta}$$

Proposition 1.3.3 *Q is diffeomorphic to the homogeneous space $SU(p+1, q+1)/H$.*

Proof: It suffices to show that

$$\begin{aligned}\eta \ :\ & SU(p+1,q+1)\times Q \mapsto Q \\ & (a_B^A, z^\alpha, w) \mapsto \left(\frac{a_0^\alpha + a_1^\alpha z^1 + \ldots + a_n^\alpha z^n + a_{n+1}^\alpha w}{a_0^0 + a_1^0 z^1 + \ldots + a_n^0 z^n + a_{n+1}^0 w}, \frac{a_0^{n+1} + a_1^{n+1} z^1 + \ldots + a_n^{n+1} z^n + a_{n+1}^{n+1} w}{a_0^0 + a_1^0 z^1 + \ldots + a_n^0 z^n + a_{n+1}^0 w}\right)\end{aligned}$$

is a transitive action on Q on the left. Let Z_0, $Z \in Q$, $\tilde Z_0$, $\tilde Z$ be the corresponding points in $\tilde Q$. Let Z_0^B, Z^A be Q-frames with $\tilde Z_0$, $\tilde Z$ as the first row. Then there exist $a_B^A \in SU(p+1, q+1)$ such that

$$a_B^A Z_0^B = Z^A$$

which shows η is transitive on Q. $\square$

$SU(p+1,q+1) \subset GL(n+2,\mathbb{C})$ being a Lie group, its Maurer-Cartan forms π_A^B are given by the equations:

$$dZ_A = \pi_A^B Z_B \tag{1.8}$$

By differentiating $(Z_A, Z_B) = h_{AB}$, one has

$$\pi_{A\bar B} + \pi_{\bar B A}$$

where the lowering of indices is relative to $h_{A\bar{B}}$. Explicitly, the equations are:

$$\begin{aligned}
&\pi_{\alpha\bar{\beta}} + \pi_{\bar{\beta}\alpha} = 0 \\
&\pi_0^{n+1} - \bar{\pi}_0^{n+1} = \pi_{n+1}^0 - \bar{\pi}_{n+1}^0 = 0 \\
&\bar{\pi}_0^0 + \pi_{n+1}^{n+1} = 0 \\
&\tfrac{i}{2}\bar{\pi}_\alpha^0 + \pi_{n+1}^\beta h_{\beta\bar{\alpha}} = 0 \\
&\bar{\pi}_\alpha^{n+1} + 2i\pi_0^\beta h_{\beta\bar{\alpha}} = 0
\end{aligned} \tag{1.9}$$

By differentiating $det(Z_0, Z_1, ..., Z_{n+1}) = 1$, one has

$$\begin{aligned}
& det(dZ_0, Z_1, ..., Z_{n+1}) + ... + det(Z_0, Z_1, ..., dZ_{n+1}) = 1 \\
\Rightarrow \quad & \pi_A^A = 0
\end{aligned}$$

By the exterior differentiation of (1.8), one obtains the structure equation of $SU(p+1, q+1)$:

$$\begin{aligned}
0 = ddZ_A &= d\pi_A^B Z_B - \pi_A^B dZ_B \\
&= d\pi_A^B Z_B - \pi_A^B \wedge \pi_B^C dZ_C \\
&= d\pi_A^C Z_C - \pi_A^B \wedge \pi_B^C dZ_C
\end{aligned}$$

Hence we have

$$d\pi_A^B = \pi_A^C \wedge \pi_C^B \tag{1.10}$$

Chapter 2

Normal Forms

As we have seen in Chapter 1, real hypersurfaces in $\mathbb{C}^n$ are in general not biholomorphically equivalent. This poses a question of finding invariants distinguishing the biholomorphism classes of real hypersurfaces. Chern and Moser [CM] solved the problem by considering "normal forms" of defining functions. The main idea is to find biholomorphisms so that certain Talyor expansion coefficients of the resulting defining function vanish. The resulting hypersurface is said to be in nomral form. Two hypersurfaces are equivalent if they have the same normal form.

In section 1 we only consider formal power series disregarding the convergence problem, which will be discussed in section 2.

In what follows we only consider non-degenerate real hypersurface in $\mathbb{C}^n$ and all biholomorphisms are local, the origin being the point of reference.

2.1 Formal theory of normal forms

Lemma 2.1.1 *Let $M = \{(z,w) \in \mathbb{C}^{n+1} | r(z,\bar{z},w,\bar{w}) = 0\}$ where r is a real analytic function with r(0)=0 and dr(0) $\neq$ 0. Then there exists a linear transfor-*

mation taking M to the form

$$\{(z^*, w^* = u^* + iv^*) \in \mathbb{C}^{n+1} | v^* = F^*(z^*, \bar{z}^*, u^*)\}$$

where F is real analytic, $F(0) = 0$ and $dF(0) = 0$.

Proof: Let

$$\begin{pmatrix} z^\alpha \\ \omega \end{pmatrix} = \begin{pmatrix} C^\alpha_\beta & C_\alpha \\ C_\beta & C \end{pmatrix} \begin{pmatrix} z^{*\beta} \\ w^* \end{pmatrix}$$

be a linear coordinate change. Then one has

$$r^*(z^*, \bar{z}^*, w^*, \bar{w}^*) =: r(C^\alpha_\beta z^{*\beta} + C_\alpha w^*, \overline{C^\alpha_\beta z^{*\beta} + C_\alpha w^*}, C_\beta z^{*\beta} + Cw^*, \overline{C_\beta z^{*\beta} + Cw^*}) = 0$$

By the chain rule,

$$\begin{aligned} r^*_{z^{*\alpha}} &= r_{z^\beta} \frac{\partial z^\beta}{\partial z^{*\alpha}} + r_w \frac{\partial w}{\partial z^{*\alpha}} \\ &= r_{z^\beta} C^\beta_\alpha + r_w C_\alpha \end{aligned}$$

$$\begin{aligned} r^*_{w^*} &= r_{z^\beta} \frac{\partial z^\beta}{\partial w^*} + r_w \frac{\partial w}{\partial w^*} \\ &= r_{z^\beta} C^\beta + r_w C \end{aligned}$$

$$\Rightarrow \begin{pmatrix} C^\beta_\alpha & C_\alpha \\ C_\alpha & C \end{pmatrix} \begin{pmatrix} r_{z^\beta}(0) \\ r_w(0) \end{pmatrix} = \begin{pmatrix} r^*_{z^{*\alpha}}(0) \\ r^*_{w^*}(0) \end{pmatrix}$$

Since $dr(0) \neq 0$, $(r_{z^\beta}(0), r_w(0))$ are not all zero, one can find c's such that

$$\begin{pmatrix} C^\beta_\alpha & C_\alpha \\ C_\alpha & C \end{pmatrix} \begin{pmatrix} r_{z^\beta}(0) \\ r_w(0) \end{pmatrix} = \begin{pmatrix} r^*_{z^{*\alpha}}(0) \\ r^*_{w^*}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}$$

so that $r^*_{u^*}(0) = 0$ and $r^*_{v^*}(0) = -2 \neq 0$. By the implicit function theorem for real analytic functions, there exists a real analytic function F^* such that $v^* = F^*(z^*, \bar{z}^*, u^*)$ and $F^*(0) = 0$. Besides, by differentiating

$$r^*(z^*, \bar{z}^*, u^* + iF^*, u^* - iF^*) = 0$$

with respect to $z^{*\alpha}$ and evaluating at the origin, one has

$$r^*_{z^{*\alpha}}(0) + r^*_{w^*}(0)(iF^*_{z^{*\alpha}}(0)) + r^*_{\bar{w}^*}(0)(-iF^*_{z^{*\alpha}}(0)) = 0$$
$$\Rightarrow \quad F^*_{z^{*\alpha}}(0) = 0$$

Similarly,

$$r^*_{w^*}(0)(1 + iF^*_{u^*}(0)) + r^*_{\bar{w}^*}(0)(1 - iF^*_{u^*}(0)) = 0$$
$$\Rightarrow \quad F^*_{u^*}(0) = 0$$

□

By virtue of the lemma we limit ourselves to hypersurfaces of the form $v = F(z, \bar{z}, u)$ where F is real analytic function in the 2n+1 variables z, $\bar{z}$, u which vanishes at the origin together with its first derivatives. From now on the condition that F is real analytic is dropped. Instead F is considered a formal power series in $z^1, \dots, z^n, \bar{z}^1, \dots, \bar{z}^n$ and u with the reality condition

$$\overline{F(z, \bar{z}, u)} = F(\bar{z}, z, u)$$

This hypersurface is subjected to a holomorphic transformation

$$z^* = f(z, w) \quad w^* = g(z, w) \tag{2.1}$$

where f is n-vector valued holomorphic and g is a holomorphic scalar. Moreover, f, g are required to preserve the origin and the complex tangent space at the origin: w=0. Thus one require

$$f(0,0) = 0 \quad g(0,0) = 0 \quad \frac{\partial g}{\partial z}(0,0) = 0$$

For the following it is useful to decompose F into semihomogeneous parts:

$$F = \sum_{\nu \geq 2} F_\nu(z, \bar{z}, u)$$

where $F_\nu(tz, t\bar{z}, t^2u) = t^\nu F_\nu(z, \bar{z}, u)$ for any $t > 0$. We call ν the "weight" of the polynomial F_ν.

Lemma 2.1.2 *There exists a biholomorphism such that the resulting F_2 is of the form $\langle z, z\rangle$ where $\langle z, z\rangle$ is a hermitian form.*

Proof: Since F contains no linear terms, one has

$$F_2 = Q(z) + \overline{Q(z)} + \langle z, z\rangle$$

where Q is a quadratic form and $\langle z, z\rangle$ is a hermitian form. Consider the transformation

$$\begin{aligned} z &= z^* \\ w &= w^* + 2iQ(z^*) \end{aligned}$$

$$\Rightarrow v = v^* + Q(z^*) + \overline{Q(z^*)} \tag{2.2}$$

On the other hand,

$$\begin{aligned} v &= Q(z) + \overline{Q(z)} + \langle z, z\rangle + \sum_{\nu\geq 3} F_\nu(z, \bar{z}, u) \\ &= Q(z^*) + \overline{Q(z^*)} + \langle z^*, z^*\rangle + \sum_{\nu\geq 3} F_\nu(z^*, \bar{z}^*, u^* + 2Re(iQ(z^*))) \end{aligned}$$

$$= Q(z^*) + \overline{Q(z^*)} + \langle z^*, z^*\rangle + \sum_{\nu\geq 3} F^*_\nu(z^*, \bar{z}^*, u^*) \tag{2.3}$$

Combining (2.2) and (2.3) the result follows. □

With this simplification one can assume M be represented by

$$v = \langle z, z\rangle + \sum_{\nu\geq 3} F_\nu(z, \bar{z}, u) \tag{2.4}$$

From now on the transformation (2.1) is restricted by the additional requirement that $\partial^2 g/\partial z^\alpha \partial z^\beta$ vanishes at the origin and we consider formal transformations of the form

$$z* = z + f = z + \sum_{\nu=2}^{\infty} f_\nu \quad w^* = w + g = w + \sum_{\nu=3}^{\infty} g_\nu \tag{2.5}$$

where $$f_\nu(tz, t^2w) = t^\nu f_\nu(z,w), \quad g_\nu(tz, t^2w) = t^\nu g_\nu(z,w)$$
The group of all formal transformations perserving the family of formal hypersurfaces of the form (2.4) as well as the origin is denoted by $\mathcal{G}_1$. One can show that the elements of $\mathcal{G}_1$ are of the form

$$z^* = Cz + \sum_{\nu=2}^{\infty} f_\nu, \quad w^* = \rho w + \sum_{\nu=3}^{\infty} g_\nu$$

where $\langle Cz, Cz\rangle = \rho\langle z, z\rangle$. Using the form (1.7) one can show that any $\phi \in G$ can be factored uniquely as

$$\phi = \psi \circ \phi_o$$

with $\phi_o \in H$ and ψ a formal transformation of the form (2.5) with

$$f_2(0,w) = 0, \quad Re\frac{\partial^2}{\partial w^2} g_4(0,w) = 0 \quad \text{at} \quad w = 0$$

The first term can be normalized by the choice of a^α in (1.7) and the second by $Re(t^{-1}\tau)$. Thus the normalization conditions for ψ are:

$$\begin{array}{lllll} f, & \frac{\partial}{\partial z^\alpha} f, & \frac{\partial}{\partial \omega} f & & \\ g, & \frac{\partial}{\partial z^\alpha} g, & \frac{\partial}{\partial \omega} g, & \frac{\partial^2 g}{\partial z^\alpha \partial z^\beta}, & Re\left(\frac{\partial^2 g}{\partial w^2}\right) \end{array} \tag{2.6}$$

all have no constant terms.

In addition to the weight decomposition, a decomposition by "type" will be introduced in the following. F is ordered in terms of powers of $z, \bar{z}$ with coefficients being power series in u. Thus

$$F = \sum_{k,l\geq 0} F_{kl}$$

where $$F_{kl}(\lambda z, \mu \bar{z}, u) = \lambda^k \mu^l F_{kl}(z, \bar{z}, u)$$
for all complex functions λ, μ and (k,l) is called the "type" of F_{kl}.

Definition 2.1.3 *Let $\langle z, z\rangle$ be written $h_{\alpha\bar{\beta}} z^\alpha z^{\bar{\beta}}$ where $\overline{h_{\alpha\bar{\beta}}} = h_{\beta\bar{\alpha}}$. Let*

$$F_{kl} = \sum a_{\alpha_1\cdots\alpha_k\bar{\beta}_1\cdots\bar{\beta}_l} z^{\alpha_1} \cdots z^{\alpha_k} \bar{z}^{\bar{\beta}_1} \cdots \bar{z}^{\bar{\beta}_l}$$

where $a_{\alpha_1\cdots\alpha_k\bar{\beta}_1\cdots\bar{\beta}_l}$ are unchanged under permutation of $\alpha_1\cdots\alpha_k$ and $\bar{\beta}_1\cdots\bar{\beta}_l$. The contraction $tr(F_{kl}) = G_{k-1,l-1}$ of F_{kl} is defined by

$$tr(F_{kl}) = \sum b_{\alpha_1\cdots\alpha_{k-1}\bar{\beta}_1\cdots\bar{\beta}_{l-1}} z^{\alpha_1}\cdots z^{\alpha_{k-1}}\bar{z}^{\bar{\beta}_1}\cdots\bar{z}^{\bar{\beta}_{l-1}}$$

where

$$b_{\alpha_1\cdots\alpha_{k-1}\bar{\beta}_1\cdots\bar{\beta}_{l-1}} = \sum h^{\alpha_k\bar{\beta}_l} a_{\alpha_1\cdots\alpha_k\bar{\beta}_1\cdots\bar{\beta}_l}$$

Here $h^{\alpha\bar{\beta}}$ is defined by

$$h^{\alpha\bar{\beta}}h_{\gamma\bar{\beta}} = \delta^{\alpha}_{\gamma}$$

Definition 2.1.4

(i) The set of admissible formal power series is

$$\mathcal{F} = \{F(z,\bar{z},u) \mid F \text{ has no term of weight less than } 3\}$$

(ii) The set of admissible formal hypersurfaces is

$$\mathcal{H} = \{v = \langle z,z\rangle + F, F \in \mathcal{F}\}$$

(iii) The set of formal normalizations is

$$\mathcal{N} = \{N \in \mathcal{F} \mid N_{kl} = 0 \; min(k,l) \le 1, trN_{22} = (tr)^2N_{32} = (tr)^3N_{33} = 0\}$$

(iv) The set of all admissible formal transformation functions is

$$\mathcal{O} = \{h = (f,g) \mid f,g \text{ are formal power series which satisfy (2.6)}\}$$

Theorem 2.1.5 *A formal hypersurface $M \in \mathcal{H}$ can be transformed by a formal transformation*

$$z^* = z + f(z,w) \quad w^* = w + g(z,w)$$

where $(f,g) \in \mathcal{O}$ into a normal form

$$v^* = \langle z,z\rangle + N \text{ with } N \in \mathcal{N}.$$

Moreover, this transformation is unique.

The proof of theorem 2.1.5 is based on a study of the following operator:

Definition 2.1.6 *The operator $L : \mathcal{O} \mapsto \mathcal{F}$ is defined by*

$$L(f,g) = Re\{2\langle z, f\rangle + ig\}_{w=u+i\langle z,z\rangle}$$

Definition 2.1.7 *A linear subspace $\tilde{\mathcal{N}}$ of $\mathcal{F}$ is called a complement of $L\mathcal{O}$ if*

$$\mathcal{F} = L\mathcal{O} \oplus \tilde{\mathcal{N}} \quad and \quad \tilde{\mathcal{N}} \cap L\mathcal{O} = \{0\}$$

Lemma 2.1.8 *Let $\tilde{\mathcal{N}}$ be a complement of $L\mathcal{O}$. A formal hypersurface $M \in \mathcal{H}$ can be transformed by a unique formal transformation*

$$z^* = z + f(z,w) \quad w^* = w + g(z,w)$$

where $(f,g) \in \mathcal{O}$ into a hypersurface of the form

$$v^* = \langle z, z\rangle + N \text{ with } N \in \tilde{\mathcal{N}}.$$

Proof: Consider $z^* = z + f$, $w^* = w + g$ where $h = (f,g) \in \mathcal{O}$. One wants $v^* = \langle z^*, z^*\rangle + F^*$, $F^* \in \tilde{\mathcal{N}}$ so one works backward to find the transformation rule.

$$\begin{aligned} & v + Img = \langle z+f, z+f\rangle + F^*(z+f, \bar{z}+\bar{f}, u+Reg) \\ \Rightarrow\ & \langle z,z\rangle + F + Img = \langle z,z\rangle + \langle z,f\rangle + \langle f,z\rangle + \langle f,f\rangle + F^*(z+f,\bar{z}+\bar{f},u+Reg) \end{aligned}$$

$$\Rightarrow Re(2\langle z,f\rangle + ig) = F(z,\bar{z},u) - F^*(z+f,\bar{z}+\bar{f},u+Reg) - \langle f,f\rangle \tag{2.7}$$

Expand f, g as follows:

$$f\left(z, u + i(\langle z,z\rangle + F)\right) = f + f^{'}iF + f^{''}\frac{(iF)^2}{2!} + \cdots \tag{2.8}$$

where the argument of f, f'... are $(z, u + i\langle z, z\rangle)$ and the prime indicates differentiation with respect to u.

By substituting (2.8) into (2.7) and collecting the terms in weight 3 one has

$$L(f_2, g_3) = F_3(z, \bar{z}, u) - F_3^*(z, \bar{z}, u) \tag{2.9}$$

where $\mathcal{F}_\nu$ denotes the set of $F \in \mathcal{F}$ of weight ν. Since $F_\nu = L(\mathcal{O})_\nu \oplus \tilde{\mathcal{N}}_\nu$ for all $\nu \geq 3$, for $F_3 \in \mathcal{F}_3$ there exists a unique (f_2, g_3) and a unique $F_3^* \in \tilde{\mathcal{N}}_3$ such that (2.9) is satisfied.

Similarly, by collecting terms of weight μ one has

$$L(f_{\mu-1}, g_\mu) = F_\mu - F_\mu^* + \cdots \tag{2.10}$$

where the dots indicate terms depending on $f_{\nu-1}$, g_ν, F_ν, F_ν^* with $\nu < \mu$. One can require that F_ν^* belongs to $\tilde{\mathcal{N}}$ and solve the resulting equation for h. By using induction (f,g) can be determined such that the function F^* belongs to $\tilde{\mathcal{N}}$. □

For the description of a complement of $L\mathcal{O}$ one decomposes $\mathcal{F}$ as

$$\mathcal{F} = \mathcal{R} + \mathcal{N}$$

where $\mathcal{R}$ consists of series of the type

$$\mathcal{R} = \sum_{min(k,l)\leq 1} R_{kl} + G_{11}\langle z, z\rangle + (G_{10} + G_{01})\langle z, z\rangle^2 + G_{00}\langle z, z\rangle^3$$

where G_{jm} is of type (j, m) and $\mathcal{N}$ is given by definition 2.1.4. This constitutes a decomposition of $\mathcal{F}$, i.e. any F can be uniquely written as $F = R + N$ with $R \in \mathcal{R}$ and $N \in \mathcal{N}$. Thus $PF = R$ defines a projection operator with range $\mathcal{R}$ and null space $\mathcal{N}$. One computes that

$$PF = \sum_{min(k,l)\leq 1} F_{kl} + G_{11}\langle z, z\rangle + (G_{10} + G_{01})\langle z, z\rangle^2 + G_{00}\langle z, z\rangle^3$$

where

$$G_{11} = \frac{4}{n+2}tr(F_{22}) - \frac{2}{(n+1)(n+2)}(tr)^2(F_{22})\langle z, z\rangle$$
$$G_{10} = \frac{6}{(n+1)(n+2)}(tr)^2F_{32}$$
$$G_{00} = \frac{6}{n(n+1)(n+2)}(tr)^3F_{33}$$

Lemma 2.1.9 *L maps $\mathcal{O}$ one to one onto $\mathcal{R} = P\mathcal{F}$.*

The proof of theorem 2.1.5 is completed by showing that $\mathcal{N}$ represents a complement of $L\mathcal{O}$. This can be concluded by lemma 2.1.9.

Proof: It suffices to show

$$Lh = F \quad mod\, \mathcal{N}$$

has a unique solution $h \in \mathcal{O}$. Collecting terms of equal type one has to solve the equations

$$(Lh)_{kl} = F_{kl} \quad \text{for } min(k,l) \leq 1$$
$$(Lh)_{kl} = F_{kl} \quad mod\, \mathcal{N} \quad \text{for } (k,l) = (2,2), (3,2), (3,3)$$

Since F is real one only consider $k \geq l$. By using the identity

$$f(z, u + i\langle z, z\rangle) = \sum_{\nu=1}^{\infty} \left(\frac{\partial}{\partial\omega}\right)^{\nu} f(z,u)\frac{i^{\nu}\langle z, z\rangle^{\nu}}{\nu!}$$

and expanding $f(z,w)$, $g(z,w)$ in power of $z, \bar{z}$

$$f = \sum_{k=0}^{\infty} f_k \quad g = \sum_{k=0}^{\infty} g_k$$

where $f_k(tz,w) = t^k f_k(z,w)$,$g_k(tz,w) = t^k g_k(z,w)$, one writes Lh in the form

$$\begin{aligned} Lh &= Re\{2\langle f, z\rangle + ig\}_{w=u+i\langle z,z\rangle} \\ &= \langle f + f'i\langle z, z\rangle + ..., z\rangle + \frac{i}{2}(g + g'i\langle z, z\rangle + ...) + \text{complex conj.} \end{aligned}$$

where the arguments of f, f', ..., g, g', ... are z, u and the prime indicates differentiation with respect to u. Collecting terms of equal type (k, l), one has, for $k \geq 2$

$$\begin{aligned} &ig_k = 2F_{k0} \\ &2\langle f_{k+1}, z\rangle - g_k'\langle z, z\rangle = 2F_{k+1,1} \end{aligned} \tag{2.11}$$

for $k = 1$,

$$\begin{aligned} &ig_1 + 2\langle z, f_0\rangle = 2F_{10} \\ &-g_1'\langle z, z\rangle + 2\langle f_2, z\rangle - 2i\langle z, f_0'\rangle\langle z, z\rangle = 2F_{21} \\ &-\tfrac{i}{2}g_1''\langle z, z\rangle^2 + 2i\langle f_2', z\rangle\langle z, z\rangle - \langle z, f_0''\rangle\langle z, z\rangle^2 = 2F_{32} \quad mod\,\mathcal{N} \end{aligned} \tag{2.12}$$

for $k = 0$,

$$\begin{aligned} &-Img_0 = F_{00} \\ &\tfrac{1}{2}Img_0''\langle z, z\rangle^2 - 2Im\langle f_1', z\rangle\langle z, z\rangle = F_{22} \quad mod\,\mathcal{N} \\ &-Reg_0'\langle z, z\rangle + 2Re\langle f_1, z\rangle = F_{11} \\ &\tfrac{1}{6}Reg_0'''\langle z, z\rangle^3 - Re\langle f_1'', z\rangle\langle z, z\rangle^2 = F_{33} \quad mod\,\mathcal{N} \end{aligned} \tag{2.13}$$

One can solve f_{k+1}, g_k uniquely for $k \geq 2$ from equations (2.11). Equations (2.12) are equivalent to

$$\begin{aligned} &ig_1 + 2\langle z, f_0\rangle = 2F_{10} \\ &-g_1'\langle z, z\rangle + 2\langle f_2, z\rangle - 2i\langle z, f_0'\rangle\langle z, z\rangle = 2F_{21} \\ &-4\langle z, f_0''\rangle\langle z, z\rangle^2 = 2F_{32} - 2iF_{21}'\langle z, z\rangle - F_{10}''\langle z, z\rangle^2 \quad mod\,\mathcal{N} \end{aligned}$$

Since the third equation has to be solved (mod $\mathcal{N}$) only one can replace the right-hand side by its projection into $\mathcal{R}$, which is called $G_{10}\langle z, z\rangle$ so that

$$-4\langle z, f_0''\rangle = G_{10}$$

Here f_0 is fixed up to a linear function in w. But by the normalization $f(0,0) = \frac{\partial f(0,0)}{\partial \omega} = 0$, f_0 is uniquely determined. g_1 and f_2 are uniquely determined by the first and second equations accordingly.

Finally one has to solve (2.13). Since

$$F_{22} = G_{11}\langle z, z\rangle + N_{22}, N_{22} \in \mathcal{N}$$

the second equation is equivalent to

$$\frac{1}{2}Img_0''\langle z,z\rangle - 2Im\langle f_1',z\rangle = G_{11}$$

which can be solved with the first for Img_o and $Im\langle f',z\rangle = (d/du)Im\langle f_1,z\rangle$. Since $\frac{\partial}{\partial z^\alpha}f$ has no constant terms f_1 vanishes for $u=0$, Img_o, $Im\langle f_1,z\rangle$ can be determined uniquely by the normalization $g(0,0)=0$ and $\frac{\partial g}{\partial w}(0,0)=0$.

Finally, since

$$F_{33} + \frac{1}{2}F_{11}''\langle z,z\rangle^2 = G_{00}\langle z,z\rangle^3 \quad mod\, \mathcal{N}$$

The last two equations of (2.14) is equivalent to

$$\begin{aligned} &-Reg_o'\langle z,z\rangle + 2Re\langle f_1,z\rangle = F_{11} \\ &-\frac{1}{3}Reg_o''' = G_{oo} \end{aligned}$$

The last equation can be solved for Reg_o''' and then the first for $Re\langle f_1,z\rangle$. By the normalization $g(0,0) = \frac{\partial g}{\partial w}(0,0) = Re\frac{\partial^2 g}{\partial w^2}(0,0) = 0$, Reg_o and $Re\langle f_1,z\rangle$ are uniquely determined. $\square$

2.2 Geometric theory of normal forms

In the last section we study that a formal hypersurface in $\mathbb{C}^{n+1}$ can be transformed to a formal normal form by a formal transformation. Now we study that for real analytic hypersurfaces the formal transformations are actually convergent and correspond to a geometric structure on the hypersurfaces.

The following notations will be adopted throughout this section: Let M be a real analytic hypersurface in $\mathbb{C}^{n+1}$ and γ a real analytic curve on M which is transversal to the complex tangent space of M. Let $\{e_\alpha\}$ be a frame of linear independent vectors on $T_{\mathbb{C}}$ which is real analytic along γ. γ and $\{e_\alpha\}$ are given locally on a distinguished point p on γ.

Theorem 2.2.1 *Let M, γ, $\{e_\alpha\}$ and p be as above. There exists a unique holomorphic mapping taking*

(i) M to the form $F_{11}(z,\bar{z},u) + \sum_{min(k,l)\geq 2} F_{kl}(z,\bar{z},u)$

(ii) p to the origin $z = w = 0$

(iii) $\{e_\alpha\}$ to $\phi_*(e_\alpha) = \frac{\partial}{\partial z^\alpha}$

(iv) γ to the curve $z = 0$, $w = \xi$, ξ is a real parameter ranging over an interval.

Assume coordinates be introduced so that p is the origin and the complex tangent space of M at p is given by $\{(z^\alpha, w) \in \mathbb{C}^{n+1} | w = 0\}$. Let γ be given by

$$z = p(\xi) \quad p(0) = q(0) = 0$$
$$w = q(\xi) \quad q^{'}(0) \neq 0$$

Consider the holomorphic mapping

$$z = p(w^*) + z^*$$
$$w = q(w^*)$$

Since

$$det \begin{pmatrix} \frac{\partial z^\alpha}{\partial z^{*\beta}} & \frac{\partial z^\alpha}{\partial w^*} \\ \frac{\partial w}{\partial z^{*\beta}} & \frac{\partial w}{\partial w^*} \end{pmatrix}_{(0,0)} = det \begin{pmatrix} I & \frac{\partial p(0,0)}{\partial w^*} \\ 0 & q'(0) \end{pmatrix} \neq 0$$

Its inverse exists near the origin and takes γ to the curve $z = 0$, $w = \xi$. Therefore one can assume M is given by $v = F(z, \bar{z}, u)$ and γ by the curve $z = 0$, $w = \xi$ so that $F(0, 0, u) = 0$.

The proof of theorem 2.2.1 is divided into three lemmas. In the following we let

$F^\omega = \{F(z, \bar{z}, u) | F$ is real analytic in some neigborhood of the origin and $F(0, 0, 0) = 0\}$

Lemma 2.2.2 *If $F \in F^\omega$ and $F(0, 0, u) = 0$ then there exists a unique holomorphic mapping*

$$z^* = z; \quad w^* = w + g(z, w); \quad g(0, w) = 0$$

taking $v = F(z, \bar{z}, u)$ to $v^* = F^*(z^*, \bar{z}^*, u^*)$ where $F^*_{k0} = F^*_{0k} = 0$ for $k = 1, 2, \ldots$

Proof: It suffices to find g such that $F^*(z^*, 0, u) = 0$, which implies $F^*(0, \bar{z}^*, u^*) = 0$. A condition for g is given by

$$F^*(z^*, \bar{z}^*, u^*) = \frac{1}{2i}(g(z, w) - \overline{g(z, w)}) + F(z, \bar{z}, u) \tag{2.14}$$

where $u^* = u + \frac{1}{2}(g(z, w) + \overline{g(z, w)})$, $\quad w = u + iF(z, \bar{z}, u)$. Regard $z, \bar{z}$ as independent variables and set $\bar{z} = 0$ in (2.14) one has

$$0 = F^*(z^*, 0, u^*) = \frac{1}{2i} g(z, u + iF(z, 0, u)) + F(z, 0, u) \tag{2.15}$$

To solve (2.15) one set

$$s = u + iF(z, 0, u)$$

Since $\frac{\partial s}{\partial u}\big|_{(0,0)} = 1 + iF_u(0, 0, 0) = 1 \neq 0$, by the implicit function theorem one can solve for u

$$u = s + G(z, s), \quad G(0, s) = 0$$

The equation (2.15) can be written $0 = \frac{1}{2i}g(z,s) + \frac{1}{i}(s-u)$ or $u = s + \frac{1}{2}g(z,s)$. Thus $g(z,w) = 2G(z,w)$ is the desired solution which vanishes for $z = 0$. The lemma is proved by the reversing the above steps. □

Lemma 2.2.3 *Let $F \in F^\omega$ and $F_{k0} = F_{0k} = 0$ for $k = 0, 1, 2, ...$ and $F_{11}(z, \bar{z}, 0)$ nondegenerate then there exists a holomorphic mapping*

$$z^* = z + f(z,w); \quad w^* = w \tag{2.16}$$

with $f(0,w) = f_z(0,w) = 0$ and such that $v = F(z,\bar{z},u)$ is mapped into

$$v^* = F_{11}^*(z^*, \bar{z}^*, u^*) + \sum_{min(k,l)\geq 2} F_{kl}^*$$

Proof: Let $\mathcal{O}_{\kappa\lambda}$ be the set of power series in $z, \bar{z}$ containing only terms of type (k,l) with $k \geq \kappa$, $l \geq \lambda$ and $\mathcal{O}^*_{\kappa\lambda}$ be the set of power series in $z^*, \bar{z}^*$ containing only terms of type (k,l) with $k \geq \kappa$, $l \geq \lambda$.
Since $F_{k0} = F_{0k} = 0$, $F(z,\bar{z},u)$ can be written

$$F(z,\bar{z},u) = F_{11}(z,\bar{z},u) + z^\alpha A_\alpha(\bar{z},u) + z^{\bar{\alpha}}\overline{A_\alpha(\bar{z},u)} + \mathcal{O}_{22}$$

where $A_\alpha(\bar{z},u) = \mathcal{O}_{02}$. Since $F_{11}(z,\bar{z},0)$ is nondegenerate one can restrict u to a small interval such that the Levi form

$$F_{11}(z,\bar{z},u) = \sum h_{\alpha\bar{\beta}}(u) z^\alpha z^{\bar{\beta}}$$

is also nondegenerate. Let $(h^{\alpha\bar{\beta}})$ be the inverse of $(h_{\alpha\bar{\beta}})$ and the holomorphic vector function is defined by

$$\overline{f^\beta(z,u)} = h^{\alpha\bar{\beta}}(u) A_\alpha(\bar{z},u) = \mathcal{O}_{02}$$

Since $f(z,u) \in \mathcal{O}_{20}$, one has $f(0,u) = f_z(0,u) = 0$ which imply $f(0,w) = f_z(0,w) = 0$. Then

$$\begin{aligned} & F_{11}(z+f, \bar{z}+\bar{f}, u) \\ = \;& F_{11}(z,\bar{z},u) + F_{11}(z,\bar{f},u) + F_{11}(f,\bar{z},u) + F_{11}(f,\bar{f},u) \\ = \;& F_{11}(z,\bar{z},u) + h_{\alpha\bar{\beta}} z^\alpha f^{\bar{\beta}} + h_{\alpha\bar{\beta}} f^\alpha z^{\bar{\beta}} + h_{\alpha\bar{\beta}} f^\alpha f^{\bar{\beta}} \end{aligned}$$

$$\begin{aligned}
&= F_{11}(z,\bar z,u) + h_{\alpha\bar\beta}(u)z^{\alpha}h^{\gamma\bar\beta}(w)A_{\gamma}(\bar z,w) + h_{\alpha\bar\beta}(u)\overline{h^{\gamma\bar\alpha}(w)A_{\gamma}(\bar z,w)}z^{\bar\beta} + h_{\alpha\bar\beta}f^{\alpha}f^{\bar\beta} \\
&= F_{11}(z,\bar z,u) + h_{\alpha\bar\beta}(u)z^{\alpha}h^{\gamma\bar\beta}(u)A_{\gamma}(\bar z,u) + h_{\alpha\bar\beta}(u)h^{\gamma\bar\alpha}(u)z^{\bar\beta}A_{\bar\gamma}(\bar z,u) + \mathcal{O}_{22} \\
&\quad \text{(by applying the identity } \varphi(z,u+iv) = \sum_{\mu}(\frac{\partial}{\partial w})^{\mu}\varphi(z,u)\frac{(iv)^{\mu}}{\mu!} \text{ to } h^{\alpha\bar\beta} and A_{\alpha}) \\
&= F_{11}(z,\bar z,u) + z^{\alpha}A_{\alpha}(\bar z,u) + z^{\bar\beta}A_{\bar\beta}(\bar z,u) + \mathcal{O}_{22} \\
&= F(z,\bar z,u) + \mathcal{O}_{22}
\end{aligned}$$

Since

$$\begin{pmatrix} \frac{\partial z^{*\alpha}}{\partial z^{\beta}} & \frac{\partial z^{*\alpha}}{\partial w} \\ \frac{\partial w^{*}}{\partial z^{\beta}} & \frac{\partial w^{*}}{\partial w} \end{pmatrix}_{(0,0)} = I$$

$$\Rightarrow z = z^{*} + g(z^{*},w^{*}) \quad w = w^{*}$$

where g begins with second order in z^{*}, $\mathcal{O}_{22}$ is mapped to $\mathcal{O}^{*}_{22}$ by (2.16). Then

$$\begin{aligned}
v^{*} = v &= F(z,\bar z,u) \\
&= F_{11}(z+f,\bar z+\bar f,u) + \mathcal{O}_{22} \\
&= F^{*}_{11}(z^{*},\bar z^{*},u^{*}) + \mathcal{O}^{*}_{22}
\end{aligned}$$

□

Lemma 2.2.4 *There exists a unique holomorphic mapping taking the frame* $\{e_{\alpha}\}$ *to* $\frac{\partial}{\partial z^{\alpha}}$ *and preserving hypersurfaces of the form*

$$v = F_{11}(z,\bar z,u) + \sum_{min(k,l)\geq 2} F_{kl}(z,\bar z,u)$$

as well as the curve $z = 0$, $w = \xi$.

Proof: Consider the holomorphic mapping

$$z^{*} = M(w)z \quad w^{*} = w$$

where $M(w)$ is a nonsingular matrix depending holomorphically on w. One can write

$$z = N(w^*)z^* \quad w = w^*$$

Since $F(z, \bar{z}, u) = v = v^* = F^*(z^*, \bar{z}^*, u^*)$, $F(z, 0, u) = 0 \Leftrightarrow F^*(z^*, 0, u^*) = 0$. Let $F_{11}(z, \bar{z}, u) = h_{\alpha\bar{\beta}}(u)z^\alpha z^{\bar{\beta}}$, then

$$\begin{aligned}
F_{11}(z, \bar{z}, u) &= F_{11}(N(w^*)z^*, \overline{N(w^*)z^*}, u^*) \\
&= h_{\alpha\bar{\beta}}(u^*)(N^\alpha_\rho(w^*)z^{*\rho})\overline{(N^\beta_\sigma(w^*)z^{*\sigma})} \\
&= h_{\alpha\bar{\beta}}(u^*)N^\alpha_\rho(u^*)\overline{N^\beta_\sigma(u^*)}z^{*\rho}z^{*\bar{\sigma}} + \mathcal{O}^*_{22} \\
&\quad \text{(by applying the identity } \varphi(z, u + iv) = \sum_\mu (\frac{\partial}{\partial w})^\mu \varphi(z, u)\frac{(iv)^\mu}{\mu!} \text{ to } N) \\
&=: \tilde{h}(u^*)_{\rho\bar{\sigma}}z^{*\rho}z^{*\bar{\sigma}} + \mathcal{O}^*_{22}
\end{aligned}$$

Since $\mathcal{O}_{22}$ is mapped to $\mathcal{O}^*_{22}$, one has

$$\begin{aligned}
v^* &= v \\
&= F_{11}(z, \bar{z}, u) + \sum_{min(k,l)\geq 2} F_{kl}(z, \bar{z}, u) \\
&= F^*_{11}(z^*, \bar{z}^*, u^*) + \sum_{min(k,l)\geq 2} F^*_{kl}(z^*, \bar{z}^*, u^*)
\end{aligned}$$

Finally, one can choose a unique $M(w)$ such that the frame $\{e_\alpha\}$ is transformed into $\frac{\partial}{\partial z^\alpha}$. This completes the proof of the lemma and the proof of theorem 2.2.1. □

Lemma 2.2.5 *The hermitian form $F_{11}(z, \bar{z}, u)$ can be made independent of u by a unique holomorphic map.*

Proof: Consider the linear transformation

$$z^* = C(w)z \quad w^* = w$$

where $C(w)$ is determined uniquely such that

$$\begin{aligned}
F_{11}(C(u)z, \overline{C(u)z}, 0) &= F_{11}(z, \bar{z}, u) \\
\text{and } F_{11}(C(u)z, \bar{z}, 0) &= F_{11}(z, \overline{C(u)z}, 0)
\end{aligned} \tag{2.17}$$

Denoting the matrix $(h_{\alpha\bar{\beta}}(u))$ by $H(u)$ (2.17) can be written

$$\begin{aligned} C^*(u)H(0)C(u) &= H(u) \\ H(0)C(u) &= C^*(u)H(0) \end{aligned} \tag{2.18}$$

Eliminating $C^*(u)$ one obtain

$$C^2(u) = H(0)^{-1}H(u)$$

Since $H(0)^{-1}H(u)$ is close to the identity matrix for small u there exists a unique matrix $C(u)$ with $C(0) = I$. This solution depends analytically on u and satisfies (2.18). □

By the above discussion one can assume the hypersurface be given by

$$v = \langle z, z \rangle + \sum_{min(k,l)\geq 2} F_{kl}(z, \bar{z}, u) \tag{2.19}$$

and γ is given by $z = 0$, $w = \xi$. The freedom in the change of variables preserving γ and the above form of M is given by linear map $z^* = U(w)z$, $w^* = w$ which preserve the form $\langle z, z \rangle$. In other words one can prescribe an analytic frame $e_\alpha(u)$ along the u-axis which is normalized by

$$\langle e_\alpha, e_\beta \rangle = h_{\alpha\bar{\beta}} \text{ where } \langle z, z \rangle = \sum h_{\alpha\bar{\beta}} z^\alpha z^{\bar{\beta}}$$

The coefficients of $F_{kl}(z, \bar{z}, u)$ in (2.19) can be viewed as local functionals depending on the curve γ: $z = p(\xi)$, $w = q(\xi)$

Lemma 2.2.6 *The coefficients of F_{kl} in (2.19) depend analytically on p, q, $\bar{q}$, $\bar{q}$ and their derivatives of order $\leq k + l$. More precisely, these coefficients depend rationally on the derivatives p', $\bar{p}'$, q' etc.*

Proof: Let $v = G(z, \bar{z}, u)$ be the given hypersurfaces containing the curve $z = p(\xi)$, $w = q(\xi)$ where $Re q'(0) \neq 0$. The condition that this curve be transversal to the complex tangent space amounts to

$$Re(q' - 2iG_z p' - iG_u q')|_{\xi=0} \neq 0 \tag{2.20}$$

Subject the hypersurface to the transformation

$$z = p(w^*) + z^* \quad w = q(w^*)$$

one has

$$\frac{1}{2i}(q - \bar{q}) - G(p + z^*, \bar{p} + \bar{z}^*, \frac{1}{2}(q + \bar{q})) = 0 \tag{2.21}$$

where the arguments in p, q are w^*. Since the given curve was assumed to lie on the given hypersurface one has $v^* = 0$ as a solution of (2.21) if $z^* = \bar{z}^* = 0$.

To simplify the notation one drops the star and denotes the left-hand side of (2.21) by

$$\Phi(z, \bar{z}, u, v) = \sum_{\zeta+\nu} \Phi_{\zeta\nu}$$

where $\Phi_{\zeta\nu}$ is a polynomial in z, $\bar{z}$, v, homogeneous of degree ζ in z, $\bar{z}$ and of degree ν in v. With this notation and by the identity

$$p(z, u + iv) = \sum_{\mu} (\frac{d}{dw})^{\mu} p(z, u) \frac{(iv)^{\mu}}{\mu!}$$

, (2.21) can be written

$$Av + \Phi_{10} + \sum_{\zeta+\nu\geq 2} \Phi_{\zeta\nu} = 0 \tag{2.22}$$

where $Av = \Phi_{01} = Re\{q' - 2iG_z(p, \bar{p}, \frac{1}{2}(q+\bar{q}))p' - iG_u q'\}v$. Thus A is an analytic function of p, $\bar{p}$, q, $\bar{q}$ and their derivatives. Note that by (2.20), one has $A \neq 0$ for small $|u|$. Similarly, the coefficients of $\Phi_{\zeta\nu}$ are analytic functions of p, $\bar{p}$, q, $\bar{q}$ at $\xi = u$ and their derivatives of order $\leq \nu$.

To obtain the same property for the coefficients of F^*, we solve (2.22) for v as a power series in z, $\bar{z}$. Let

$$v = V_1 + V_2 + ...$$

where V_ζ are homogeneous polynomials in z, $\bar{z}$ of degree ζ. By comparision of coefficients in (2.22), one obtains AV_ζ as a polynomial in V_1, V_2, ..., $V_{\zeta-1}$ with

coefficients analytic in p, q, $\bar{p}$, $\bar{q}$ and their derivatives of order $\leq \zeta$. In dependence on the derivatives they are rational, the denominator being a power of A.
To complete the proof one subjects the hypersurface to the holomorphic transformation of lemmas 2.2.2, 2.2.3, 2.2.4 and 2.2.5 which preserve the curve $z = 0$, $w = \xi$. From the proofs of the lemmas, one has that the coefficients of the transformation as well as the resulting hypersurface (2.19) have the stated dependence on p, q. $\Box$

One can fix the curve γ, the associated frame $\{e_\alpha\}$ and its parametrization so that

$$trF_{22} = 0 \quad (tr)^2 F_{32} = 0 \quad (tr)^3 F_{33} = 0$$

Proposition 2.2.7 *The condition $(tr)^2 F_{32} = 0$ gives rise to a second order differential equation for the curve γ where the parametrization is ignored.*

Proof: Assume the parametrization is fixed. According to lemms 2.2.6 the coefficients of F_{32} are analytic functions of p, $\bar{p}$ and their derivatives up to order 5. We claim that if the hypersurface is in the form (2.19) then F_{32} depends on the derivatives of order ≤ 2 and is of the form

$$F_{32} = \langle z, Bp^{''} \rangle \langle z, z \rangle^2 + K_{32} \tag{2.23}$$

where K_{32}, B depend on p, $\bar{p}$, $p^{'}$, $\bar{p}^{'}$ analytically, and B is a nonsingular matrix for small $|u|$.
To prove the claim we recall that (2.19) was obtained by a transformation

$$z \to p(w) + C(w) + ... \quad w \to q(w) + ...$$

one chooses $Req(u) = u$ fixing the parametrization. $Imq(u)$ is determined by p, $\bar{p}$. To study the dependence of F_{32} at $u = u_o$ one subjects (2.19) to the transformation

$$z = s(w^*) + z^* + ..., \quad w = q(w^* + u_o)$$

This amounts to replacing $p(u)$ by $p^*(u^*) = p(u_o+u)+C(u_o+u)s(u)$. Considering p and p' fixed at $u = u_o$ one requires $s(0) = 0$ and $s^{'}(0) = 0$ and investigates the dependence of F_{32} on the germ of s at $u = u_o$. One chooses the higher order terms in the above equation in such a way that the form of (2.19) is preserved as far as terms of weight ≤ 5 is concerned. This can be done by the transformation

$$\begin{aligned} z &= z^* + s(w^*) + 2i\langle z^*, s^{'}(\bar{w}^*)\rangle z^* \\ w &= w^* + u_0 + 2i\langle z^*, s(\bar{w}^*)\rangle \end{aligned}$$

Since the hermitian form $\langle\ ,\ \rangle$ is antilinear in the second argument this transformation is holomorphic. Then one computes

$$v - \langle z, z\rangle = v^* - \langle z^*, z^*\rangle + 4Re\langle z^*, s^{''}(0)\rangle\langle z^*, z^*\rangle^2 + \cdots$$

if z, w lies on the manifold (2.19). The dots indicate terms of weight ≥ 6 in z^*, $\bar{z}^*$, u^*. Thus for $u^* = 0$ and setting $z^* = z$ one has

$$F_{32}|_{u=u_0} = F^*_{32}|_{u^*=0} + 2\langle z^*, s^{''}(0)\rangle\langle z^*, z^*\rangle^2$$

Hence F^*_{32} depends on s, $s^{'}$, $s^{''}$ only. By using

$$(C(u_0 + u)s(u))^{''} = C(u_0)s^{''}(0) \quad \text{for } u = 0$$

one has

$$F^*_{32} + 2\langle z^*, C^{-1}(u_0)p^{''}(0)\rangle\langle z^*, z^*\rangle^2$$

is independent of s. Then the claim is proved with $B(u) = -2C^{-1}(u_0)$. Thus $B(0) = -2I$ and $B(u)$ is nonsingular for small values of $|u|$.

Therefore the equation $(tr)^2F_{32} = 0$ can be written as a differential equation

$$p^{''} = Q(p, \bar{p}, p^{'}\bar{p}^{'}, u)$$

where Q is analytic. For given $p(0)$, $p^{'}(0)$ there exists a unique analytic equation $p(u)$ for sufficiently small $|u|$. Choosing the curve γ in this manner one has

$(tr)^2 F_{32} = 0$.

To show the differential equation $(tr)^2 F_{32} = 0$ is independent of the parametrization and the frame $\{e_\alpha\}$, one subjects the hypersurface (2.19) to the most general self mapping

$$\begin{aligned} z &\rightarrow g'(w)^{\frac{1}{2}} U(w) z \\ w &\rightarrow g(w) \end{aligned}$$

where $Img(u) = 0$, $g(0) = 0$, $g'(0) > 0$, $\langle Uz, Uz\rangle = \langle z, z\rangle$ for real w. Under such mapping F_{32} is replaced by

$$g'^{\frac{3}{2}} F_{32}(U^{-1}z, \bar{U}^{-1}\bar{z}, g^{-1}(u))$$

and the equation $(tr)^2 F_{32} = 0$ remains satisfied for $z = 0$. Thus $(tr)^2 F_{32}$ is a differential equation for γ irrespective of the parametrization and the frame. $\square$

Proposition 2.2.8 *The frame $\{e_\alpha\}$ can be fixed so that $trF_{22} = 0$.*

Proof: Subject (2.19) with $(tr)^2 F_{32} = 0$ to the transformation

$$\begin{aligned} z^* &= U(w) z \\ w^* &= w \end{aligned}$$

with a nonsingular matrix $U(w)$ which for $Imw = 0$ preserves the form $\langle z, z\rangle = \langle Uz, Uz\rangle$. U is defined via a differential equation

$$\frac{d}{du} U = UA \quad \text{with } \langle Az, z\rangle + \langle z, Az\rangle = 0 \tag{2.24}$$

Then

$$\begin{aligned} \langle z^*, z^*\rangle &= \langle (U + iU'\langle z, z\rangle + ...)z, (U + iU'\langle z, z\rangle + ...)z\rangle \\ &= \langle (I + iA\langle z, z\rangle + ...)z, (I + iA\langle z, z\rangle + ...)z\rangle \\ &= \langle z, z\rangle(1 + 2i\langle Az, z\rangle + ...) \end{aligned}$$

where the arguments of U, A are u and the dots indicate terms of order ≥ 6 in z, $\bar{z}$. Thus

$$F_{22}^* = F_{22} + 2i\langle Az, z\rangle\langle z, z\rangle \quad F_{32}^* = F_{32}$$

where on the left side $z^* = U(u)z$. Since trF_{22} is a hermitian form the equation $trF_{22}^* = 0$ determines $\langle iAz, z\rangle$ uniquely as a hermitian form, hence A is uniquely determined as an antihermitian matrix with respect to $\langle\ ,\ \rangle$. Thus (2.24) defines a $U(u)$, analytic in u and preserving the form $\langle\ ,\ \rangle$ if $U(0)$ does. Geometrically, this can be viewed as a first order differential equation

$$\frac{de_\alpha}{du} = \sum a_\alpha^\beta(u)e_\beta \quad \langle e_\alpha, e_\beta\rangle = h_{\alpha\bar{\beta}}$$

□

Proposition 2.2.9 *The parametrization for the curve γ can be fixed so that $(tr)^3F_{33} = 0$.*

Proof: Consider the transformation

$$z^* = q^{'}(w)^{\frac{1}{2}}z$$
$$w^* = q(w)$$

with $q(0) = 0 \quad \overline{q(w)} = q(\bar{w}) \quad q^{'}(0) > 0$. Then

$$v^* = q^{'}(u)v - \frac{1}{6}q^{'''}v^3 + ... \tag{2.25}$$

By using

$$q^{'}(w)^{\frac{1}{2}} = q^{'}(u)^{\frac{1}{2}} + \tfrac{1}{2}q^{'}(u)^{-\frac{1}{2}}q^{''}(u)iv + \tfrac{1}{2}\{-\tfrac{1}{2}q^{'}(u)^{\frac{-3}{2}}(q^{''}(u))^2 + q^{'}(u)^{-\frac{1}{2}}q^{'''}(u)\}\tfrac{(iv)^2}{2} + \cdots$$

one has

$$\langle z^*, z^*\rangle = g^{'}(u)\langle z, z\rangle - \frac{1}{2}(q^{'''} - \frac{(q^{''})^2}{q^{'}})\langle z, z\rangle^3 + \cdots \tag{2.26}$$

Substitute (2.25) into (2.26) one has

$$v^* - \langle z^*, z^* \rangle = q'(v - \langle z, z \rangle) + (\frac{1}{3}q''' - \frac{1}{2}\frac{q''^2}{q'})\langle z, z \rangle^3 + \cdots$$

$$\text{or} \quad F^*_{33} = q' F_{33} + (\frac{1}{3}q''' - \frac{1}{2}\frac{(q'')^2}{q'})\langle z, z \rangle^3$$

Thus $trF^*_{33} = 0$ gives rise to an analytic third order differential equation for the real function $q(u)$, uniquely determined by $q(0) = 0$, $q'(0) > 0$, $q''(0)$ which are assumed real. Thus a distinguished parameter ξ for the curve γ is found and it is determined up to a real projective transformation $\xi \to \xi/(\alpha\xi + \beta)$, $\beta > 0$. □

Thus a holomorphic transformation taking M into the normal form is constructed and the existence proof has been reduced to that for ordinary differential equations. The above discussion can be summarized in the following theorem:

Theorem 2.2.10 *If M is a real analytic manifold the unique formal transformation of theorem 2.1.5 taking M into a normal form and satisfying the normalization condition is given by convergent series, i.e. defines a holomorphic mapping.*

The above differential equations defines a holomorphically invariant family of a parametrized curve γ transversal to the complex tangent bundle, with a frame $\{e_\alpha\}$ propagating along γ. The parameter ξ is fixed up to a projective transformation $\xi/(\alpha\xi + \beta)$ ($\beta \neq 0$) keeping $\xi = 0$ fixed.

Definition 2.2.11 *A curve γ on a nondegenerate hypersurfaces $M \subset \mathbb{C}^{n+1}$ is called a chain if for each point $p \in \gamma$ there is some open set $U \subset M$ and some local biholomorphism Φ such that $\Phi(u)$ has the form*

$$v = \langle z, z \rangle + \sum_{min(k,l) \geq 2} F_{kl}(z, \bar{z}, u), \quad (tr)^2 F_{32} = 0$$

$$\textit{and} \quad \Phi(\gamma \cap U) \textit{ lies on the u-axis}$$

Chapter 3

Connections and Curvatures

To solve the equivalence problem for nondegenerate real hypersurfaces in $\mathbb{C}^2$, Cartan constructed a complete set of invariant connection forms. Chern and Moser generalized the solution to higher dimensional cases. The connection allows us to define an invariant family of curves, called chains, which will be discussed in the next chapter.

3.1 Solution of the equivalence problem

In this section, we study the generalization of Cartan's solution to the equivalence problem to higher dimensions by Chern and Moser.
Let M be a (2n+1) dimensional nondegenerate CR manifold with local CR structure $(\theta,\, \theta^{\alpha},\, \theta^{\bar{\alpha}})$. Let $(\tilde{\theta},\, \tilde{\theta}^{\alpha},\, \tilde{\theta}^{\bar{\alpha}})$ be another choice. Then

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\theta}^{\alpha} \\ \tilde{\theta}^{\bar{\alpha}} \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ v^{\alpha} & u^{\alpha}_{\beta} & 0 \\ v^{\bar{\alpha}} & 0 & u^{\bar{\alpha}}_{\bar{\beta}} \end{pmatrix} \begin{pmatrix} \theta \\ \theta^{\beta} \\ \theta^{\bar{\beta}} \end{pmatrix} \tag{3.1}$$

where u is real and non-zero. v^{α}, u^{α}_{β} are complex and (u^{α}_{β}) is non-singular.

Remark

A G-structure of a manifold M of dimension 2n+1 is a reduction of the group of its tangent bundle to G. If the group of all coefficient matrices of (3.1) is denoted by G, then M has a G-structure. The G-structure is called integrable if the Frobenius condition is satisfied: $d\theta$, $d\theta^\alpha$ belong to the differential ideal generated by θ, θ^β. Since θ is real, this condition implies

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}, \quad mod\ \theta$$

where $h_{\alpha\bar{\beta}} = \bar{h}_{\beta\bar{\alpha}} = h_{\beta\bar{\alpha}}$. An integrable G-structure is called nondegenerate if $det(h_{\alpha\bar{\beta}}) \neq 0$.

Consider the half-line bundle E over M defined by

$$E = \{(x, u\theta) \mid x \in M, u > 0\}$$

The form ω given by

$$\omega(X) = u\theta(\pi_* X) \quad X \in TE_{(x,u\theta)}$$

is independent of the choice of θ, since for another choice $\tilde{\theta} = a\theta, a \neq 0$,

$$\begin{aligned}
\tilde{\omega}(\tilde{X}) &= v\tilde{\theta}(\pi_*\tilde{X}), \quad \tilde{X} \in TE_{(x,v\tilde{\theta})}, \quad v > 0, \\
&= av\theta(\pi_* X), X \in TE_{(x,av\theta)} \\
&= \omega(X)
\end{aligned}$$

Since $d\theta \equiv 0 \quad mod\ \theta, \theta^\alpha$, one has

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} \quad mod\ \theta, \quad h_{\alpha\bar{\beta}} = \overline{h_{\beta\bar{\alpha}}} = h_{\bar{\beta}\alpha}$$

$$\Rightarrow d\omega = ud\theta + du\theta = iuh_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + \omega \wedge (-\frac{du}{u} + \phi_o) \qquad (3.2)$$

where ϕ_o is a real 1 form on M. The equation (3.2) can be written

$$d\omega = ig_{\alpha\bar{\beta}}\omega^\alpha \wedge \omega^{\bar{\beta}} + \omega \wedge \phi \qquad (3.3)$$

where ω^α are linear combinations of θ^β, θ and $g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha}$ are constants.

The forms ω, $\mathrm{Re}\omega^\alpha$, $\mathrm{Im}\omega^{\bar{\alpha}}$ and ϕ constitute a basis of the cotangent space of E. The most general transformation of ω, ω^α, $\omega^{\bar{\alpha}}$ and ϕ leaving the equation (3.3) and the form ω invariant has the matrix of coefficients:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ v^\alpha & u^\alpha_\beta & 0 & 0 \\ v^{\bar{\alpha}} & 0 & u^{\bar{\alpha}}_{\bar{\beta}} & 0 \\ s & ig_{\rho\bar{\sigma}}u^\rho_\beta v^{\bar{\sigma}} & -ig_{\rho\bar{\sigma}}u^{\bar{\sigma}}_{\bar{\beta}}v^\rho & 1 \end{pmatrix} \tag{3.4}$$

where $s \in \mathbb{R}$ and $g_{\alpha\bar{\beta}}u^\alpha_\rho u^{\bar{\beta}}_{\bar{\sigma}} = g_{\rho\bar{\sigma}}$.

Let G_1 be the group of all non-singular matrices (3.4) and Y be the principal G_1-bundle over E consisting of all such form ω, ω^α, $\omega^{\bar{\alpha}}$ and ϕ. Then E has a G_1-structure and

$$\dim G_1 = (n+1)^2, \quad \dim E = 2(n+1), \quad \dim Y = \dim G_1 + \dim E = (n+2)^2 - 1$$

Now there are the canonically defined 1-forms ω, ω^α, $\omega^{\bar{\alpha}}$ and ϕ on Y and new ones will be introduced so that the total number equals the dimension of Y.

Since $d\theta^\alpha \equiv 0 \mod \theta, \theta^\beta$, one has $d\omega^\alpha \equiv 0 \mod \omega, \omega^\beta$ and

$$d\omega^\alpha = \omega^\beta \wedge \phi_{\beta\cdot}^{\ \alpha} + \omega \wedge \phi^\alpha \tag{3.5}$$

In what follows the $g_{\alpha\bar{\beta}}$'s are allowed to be variables and $g^{\alpha\bar{\beta}}$ are introduced by the equations

$$g_{\alpha\bar{\beta}}g^{\gamma\bar{\beta}} = \delta^\gamma_\alpha \quad g_{\alpha\bar{\beta}}g^{\alpha\bar{\gamma}} = \delta^{\bar{\gamma}}_{\bar{\beta}}$$

They are used to raise and lower indices and the location of an index will be indicated by a dot, for example,

$$u_{\alpha\cdot}^{\ \beta}g_{\beta\bar{\gamma}} = u_{\alpha\bar{\gamma}} \quad u^{\beta}_{\cdot\alpha}g^{\alpha\bar{\gamma}} = u^{\beta\bar{\gamma}}$$

The exterior differentiation of (3.3) gives:

$$i(dg_{\alpha\bar{\beta}} - \phi_{\gamma\cdot}^{\ \alpha}g_{\alpha\bar{\beta}} - \overline{\phi_{\beta\cdot}^{\ \gamma}g_{\gamma\bar{\alpha}}} + g_{\alpha\bar{\beta}}\phi) \wedge \omega^\alpha \wedge \omega^{\bar{\beta}} + (-d\phi + i\omega^\alpha g_{\alpha\bar{\beta}} \wedge \phi^{\bar{\beta}} - i\phi^\alpha g_{\alpha\bar{\beta}} \wedge \omega^{\bar{\beta}}) \wedge \omega = 0$$

$$\Rightarrow i(dg_{\alpha\bar\beta} - \phi_{\alpha\bar\beta} - \phi_{\bar\beta\alpha} + g_{\alpha\bar\beta}\phi) \wedge \omega^\alpha \wedge \omega^{\bar\beta} + (-d\phi + i\omega_{\bar\beta} \wedge \phi^{\bar\beta} - i\phi_{\bar\beta} \wedge \omega^{\bar\beta}) \wedge \omega = 0 \tag{3.6}$$

and the exterior differentiation of (3.5) gives:

$$(\omega^\gamma \wedge \phi_{\gamma\cdot}^{\ \beta} + \omega \wedge \phi^\beta) \wedge \phi_{\beta\cdot}^{\ \alpha} - \omega^\beta \wedge d\phi_{\beta\cdot}^{\ \alpha} + (ig_{\rho\bar\sigma}\omega^\rho \wedge \omega^{\bar\sigma} + \omega \wedge \phi) \wedge \phi^\alpha - \omega \wedge d\phi^\alpha = 0$$

$$\Rightarrow (d\phi_{\beta\cdot}^{\ \alpha} - \phi_{\beta\cdot}^{\ \gamma} \wedge \phi_{\gamma\cdot}^{\ \alpha} - i\omega_\beta \wedge \phi^\alpha) \wedge \omega^\beta + (d\phi^\alpha - \phi \wedge \phi^\alpha - \phi^\beta \wedge \phi_{\beta\cdot}^{\ \alpha}) \wedge \omega = 0 \tag{3.7}$$

Lemma 3.1.1 *There exist $\phi_{\beta\cdot}^{\ \alpha}$ which satisfy (3.5) and*

$$dg_{\alpha\bar\beta} + g_{\alpha\bar\beta}\phi - \phi_{\alpha\bar\beta} - \phi_{\bar\beta\alpha} = 0, \quad \phi_{\bar\beta\alpha} = \overline{\phi_{\beta\bar\alpha}} \tag{3.8}$$

$$or \quad dg^{\alpha\bar\beta} - g^{\alpha\bar\beta}\phi + \phi^{\alpha\bar\beta} + \phi^{\bar\beta\alpha} = 0$$

Such $\phi_{\beta\cdot}^{\ \alpha}$ are determined up to additive term in ω.

Proof: By (3.6) one has

$$dg_{\alpha\bar\beta} - \phi_{\alpha\bar\beta} - \phi_{\bar\beta\alpha} + g_{\alpha\bar\beta}\phi = A_{\alpha\bar\beta\gamma}\omega^\gamma + B_{\alpha\bar\beta\bar\gamma}\omega^{\bar\gamma} + C_{\alpha\bar\beta}\omega$$

where $A_{\alpha\bar\beta\gamma} = A_{\gamma\bar\beta\alpha}$, $B_{\alpha\bar\beta\bar\gamma} = B_{\alpha\bar\gamma\bar\beta}$. Since

$$dg_{\beta\bar\alpha} - \phi_{\beta\bar\alpha} - \phi_{\bar\beta\alpha} + g_{\beta\bar\alpha}\phi = A_{\beta\bar\alpha\gamma}\omega^\gamma + B_{\beta\bar\alpha\bar\gamma}\omega^{\bar\gamma} + C_{\beta\bar\alpha}\omega$$

$$\overline{dg_{\alpha\bar\beta}} - \overline{\phi_{\alpha\bar\beta}} - \overline{\phi_{\bar\beta\alpha}} + \overline{g_{\alpha\bar\beta}\phi} = \overline{A_{\alpha\bar\beta\gamma}\omega^\gamma} + \overline{B_{\alpha\bar\beta\bar\gamma}\omega^{\bar\gamma}} + \overline{C_{\alpha\bar\beta}\omega}$$

and from the hermitian property of $g_{\alpha\bar\beta}$, one has

$$\overline{A_{\alpha\bar\beta\gamma}} = B_{\beta\bar\alpha\bar\gamma} \quad \overline{C_{\alpha\bar\beta}} = C_{\beta\bar\alpha}$$

Let $\phi'_{\alpha\bar\beta} = \phi_{\alpha\bar\beta} + A_{\alpha\bar\beta\gamma}\omega^\gamma + \frac{1}{2}C_{\alpha\bar\beta}\omega$, then

$$\begin{aligned} & dg_{\alpha\bar\beta} - \phi'_{\alpha\bar\beta} - \phi'_{\bar\beta\alpha} + g_{\alpha\bar\beta}\phi \\ = \; & A_{\alpha\bar\beta\gamma}\omega^\gamma + B_{\alpha\bar\beta\gamma}\omega^{\bar\gamma} + C_{\alpha\bar\beta}\omega - A_{\alpha\bar\beta\gamma}\omega^\gamma - B_{\alpha\bar\beta\gamma}\omega^{\bar\gamma} - \frac{1}{2}C_{\alpha\bar\beta}\omega - \frac{1}{2}C_{\alpha\bar\beta}\omega \\ = \; & 0 \end{aligned}$$

$$\begin{aligned}
& \omega^\beta \wedge \phi'^{\alpha}_{\beta\cdot} \\
\equiv\ & \omega^\beta \wedge (\phi_{\beta\cdot}^{\alpha} + A_{\beta\cdot\gamma}^{\alpha}\omega^\gamma + \frac{1}{2}C_{\beta\cdot}^{\alpha}\omega) \\
\equiv\ & \omega^\beta \wedge \phi_{\beta\cdot}^{\alpha} \\
\equiv\ & d\omega^\alpha \quad mod\,\omega
\end{aligned}$$

Therefore $\phi'_{\alpha\bar\beta}$ satisfy (3.5) and (3.8).

Besides,

$$\begin{aligned}
0 &= g^{\gamma\bar\beta}(dg_{\alpha\bar\beta} + g_{\alpha\bar\beta}\phi - \phi'_{\alpha\bar\beta} - \phi'_{\bar\beta\alpha}) \\
&= -g_{\alpha\bar\beta}dg^{\gamma\bar\beta} + \delta^\gamma_\alpha\phi - \delta^\gamma_\mu\phi'^{\mu}_{\alpha\cdot} - g^{\gamma\bar\beta}g_{\bar\mu\alpha}\overline{\phi'^{\mu}_{\beta\cdot}} \\
&= g^{\alpha\bar\nu}(-g_{\alpha\bar\beta}dg^{\gamma\bar\beta}) + g^{\gamma\bar\nu}\phi - \phi'^{\gamma\bar\nu} - g^{\gamma\bar\beta}\delta^{\bar\mu}_{\bar\nu}\overline{\phi'^{\mu}_{\beta\cdot}} \\
&= -dg^{\gamma\bar\nu} + g^{\gamma\bar\nu}\phi - \phi'^{\gamma\bar\nu} - \phi'^{\bar\nu\gamma}
\end{aligned}$$

Finally, let $\phi_{\alpha\cdot}^{\beta}$ and $\phi'^{\beta}_{\alpha\cdot}$ both satisfy (3.5) and (3.8) and

$$\phi_{\alpha\cdot}^{\beta} = \phi'^{\beta}_{\alpha\cdot} + a_{\alpha\cdot}^{\beta}\omega + b_{\alpha\cdot\gamma}^{\beta}\omega^\gamma + c_{\alpha\cdot\bar\gamma}^{\beta}\omega^{\bar\gamma}$$

Substitute the above equation in (3.5) and (3.8) one has that

$$b_{\alpha\cdot\gamma}^{\beta} = c_{\alpha\cdot\bar\gamma}^{\beta} = 0$$

□

In what follows we will suppose (3.8) holds. Then by (3.6)

$$d\phi = i\omega_{\bar\beta} \wedge \phi^{\bar\beta} + i\phi_{\bar\beta} \wedge \omega^{\bar\beta} + \omega \wedge \psi \tag{3.9}$$

where ψ is a real one-form.

Lemma 3.1.2 *Let $\Phi_{\beta\cdot}^{\alpha}$ be exterior two-form satisfying*

$$\Phi_{\beta\cdot}^{\alpha} \wedge \omega^\beta \equiv 0, \quad \Phi_{\alpha\bar\beta} + \Phi_{\bar\beta\alpha} \equiv 0 \qquad mod\,\omega \tag{3.10}$$

Then

$$\Phi_{\alpha\bar\rho} \equiv S_{\alpha\beta\bar\rho\bar\sigma}\omega^\beta \wedge \omega^{\bar\sigma} \qquad mod\,\omega \tag{3.11}$$

where $S_{\alpha\beta\bar{\rho}\bar{\sigma}}$ has the symmetry properties:

$$S_{\alpha\beta\bar{\rho}\bar{\sigma}} = S_{\beta\alpha\bar{\rho}\bar{\sigma}} = S_{\alpha\beta\bar{\sigma}\bar{\rho}} \tag{3.12}$$

$$S_{\alpha\beta\bar{\rho}\bar{\sigma}} = \bar{S}_{\rho\sigma\bar{\alpha}\bar{\beta}} = S_{\bar{\rho}\bar{\sigma}\alpha\beta} \tag{3.13}$$

Proof: By (3.10), one has

$$\begin{aligned}
&\Phi_{\beta\bar{\alpha}} \wedge \omega^{\beta} \equiv 0 \qquad \mathit{mod}\, \omega \\
\Rightarrow\ &\Phi_{\beta\bar{\alpha}} \equiv \chi_{\beta\bar{\alpha}\gamma} \wedge \omega^{\gamma} \quad \text{where } \chi_{\beta\bar{\alpha}\gamma} \text{ are one-forms} \\
\Rightarrow\ &\Phi_{\bar{\beta}\alpha} \equiv \chi_{\bar{\beta}\alpha\bar{\gamma}} \wedge \omega^{\bar{\gamma}}
\end{aligned}$$

By the second equation of (3.10)

$$\begin{aligned}
&\chi_{\alpha\bar{\beta}\gamma} \wedge \omega^{\gamma} + \chi_{\bar{\beta}\alpha\bar{\gamma}} \wedge \omega^{\bar{\gamma}} \equiv 0 \quad \mathit{mod}\, \omega \\
\Rightarrow\ &\chi_{\alpha\bar{\beta}\gamma} \wedge \omega^{\gamma} \equiv 0 \quad \mathit{mod}\, \omega, \omega^{\bar{\sigma}} \\
\Rightarrow\ &\Phi_{\alpha\bar{\beta}} \equiv \chi_{\alpha\bar{\beta}\gamma} \wedge \omega^{\gamma} \equiv 0 \quad \mathit{mod}\, \omega, \omega^{\rho}, \omega^{\bar{\sigma}} \\
\Rightarrow\ &\Phi_{\alpha\bar{\rho}} \equiv S_{\alpha\beta\bar{\rho}\bar{\sigma}} \omega^{\beta} \wedge \omega^{\bar{\sigma}} \quad \mathit{mod}\, \omega
\end{aligned}$$

Besides,

$$\begin{aligned}
\Phi_{\alpha\bar{\rho}} &\equiv S_{\alpha\beta\bar{\rho}\bar{\sigma}} \omega^{\beta} \wedge \omega^{\bar{\sigma}} \quad \mathit{mod}\, \omega \\
\Phi_{\bar{\rho}\alpha} &= \overline{\Phi_{\rho\bar{\alpha}}} \equiv \overline{S_{\rho\sigma\bar{\alpha}\bar{\beta}}} \omega^{\bar{\sigma}} \wedge \omega^{\beta} \equiv -S_{\bar{\rho}\bar{\sigma}\alpha\beta} \omega^{\beta} \wedge \omega^{\bar{\sigma}}
\end{aligned}$$

By $\Phi_{\alpha\bar{\rho}} + \Phi_{\bar{\rho}\alpha} \equiv 0 \quad \mathit{mod}\, \omega$, one has $S_{\alpha\beta\bar{\rho}\bar{\sigma}} = S_{\bar{\rho}\bar{\sigma}\alpha\beta}$. Finally,

$$\begin{aligned}
&\Phi_{\beta\bar{\alpha}} \wedge \omega^{\beta} \equiv 0 \\
\Rightarrow\ &S_{\beta\gamma\bar{\alpha}\bar{\sigma}} \omega^{\gamma} \omega^{\bar{\sigma}} \omega^{\beta} \equiv 0 \\
\Rightarrow\ &\sum_{\beta<\gamma} (S_{\beta\gamma\bar{\alpha}\bar{\sigma}} - S_{\gamma\beta\bar{\alpha}\bar{\sigma}}) \omega^{\beta} \omega^{\bar{\sigma}} \omega^{\gamma} \equiv 0
\end{aligned}$$

one has $S_{\beta\gamma\bar{\alpha}\bar{\sigma}} = S_{\gamma\beta\bar{\alpha}\bar{\sigma}}$. The other symmetry property is obtained similarly. □

Equation (3.7) indicates the necessity of studying the expression:

$$\Pi_{\alpha\cdot}^{\ \gamma} = d\phi_{\alpha\cdot}^{\ \gamma} - \phi_{\alpha\cdot}^{\ \beta} \wedge \phi_{\beta\cdot}^{\ \gamma}$$

By (3.8)

$$\Pi_{\beta\bar{\alpha}} = g_{\gamma\bar{\alpha}} d\phi_{\beta\cdot}^{\ \gamma} - \phi_{\beta\cdot}^{\ \gamma} \wedge \phi_{\gamma\bar{\alpha}} = d\phi_{\beta\bar{\alpha}} - \phi_{\beta\bar{\alpha}} \wedge \phi - \phi_{\bar{\alpha}\gamma} \wedge \phi_{\beta\cdot}^{\ \gamma}$$

$$\Rightarrow \quad \Pi_{\beta\bar{\alpha}} + \Pi_{\bar{\alpha}\beta} = d(\phi_{\beta\bar{\alpha}} + \phi_{\bar{\alpha}\beta}) - (\phi_{\beta\bar{\alpha}} + \phi_{\bar{\alpha}\beta}) \wedge \phi$$

By using $\phi_{\beta\bar{\gamma}} \wedge \phi_{\bar{\alpha}\cdot}^{\ \bar{\gamma}} = \phi_{\beta\cdot}^{\ \gamma} \wedge \phi_{\bar{\alpha}\gamma}$ and differentiating (3.8), one has

$$\Pi_{\beta\bar{\alpha}} + \Pi_{\bar{\alpha}\beta} = g_{\beta\bar{\alpha}} d\phi \tag{3.14}$$

Define

$$\Phi_{\beta\cdot}^{\ \gamma} \equiv \Pi_{\beta\cdot}^{\ \gamma} - i\omega_\beta \wedge \phi^\gamma + i\phi_\beta \wedge \omega^\gamma + i\delta_\beta^\gamma(\phi_\sigma \wedge \omega^\sigma) \quad mod\ \omega \tag{3.15}$$

$$\Rightarrow \quad \Phi_{\beta\bar{\alpha}} \equiv \Pi_{\beta\bar{\alpha}} - i\omega_\beta \wedge \phi_{\bar{\alpha}} + i\phi_\beta \wedge \omega_{\bar{\alpha}} + ig_{\beta\bar{\alpha}}(\phi_\sigma \wedge \omega^\sigma) \quad mod\ \omega \tag{3.16}$$

By (3.7), (3.9), (3.14),

$$\Phi_{\alpha\bar{\beta}} + \Phi_{\bar{\beta}\alpha} \equiv 0 \quad mod\ \omega$$

$$\Phi_{\beta\cdot}^{\ \alpha} \wedge \omega^\beta \equiv 0 \quad mod\ \omega$$

So $\Phi_{\beta\cdot}^{\ \gamma}$ satisfy (3.11) - (3.13).

Lemma 3.1.3 *The forms $\phi_{\beta\cdot}^{\ \alpha}, \phi^\alpha, \psi$ satisfying (3.5), (3.8) and (3.9) are defined up to the transformation*

$$\phi_{\beta\cdot}^{\ \alpha} = \phi'^{\ \alpha}_{\beta\cdot} + D_{\beta\cdot}^{\ \alpha}\omega$$

$$\phi^\alpha = \phi'^\alpha + D_{\beta\cdot}^{\ \alpha}\omega^\beta + E^\alpha\omega$$

$$\psi = \psi' + G\omega + i(E_\alpha\omega^\alpha - E_{\bar{\alpha}}\omega^{\bar{\alpha}})$$

where $G \in \mathbb{R}$ and $D_{\alpha\bar{\beta}} + D_{\bar{\beta}\alpha} = 0$.

Proof: Let

$$\phi_{\beta\cdot}^{\ \alpha} = \phi'^{\ \alpha}_{\beta\cdot} + D_{\beta\cdot}^{\ \alpha}\omega$$

$$\phi^\alpha = \phi'^\alpha + E^\alpha\omega + R_{\beta\cdot}^{\ \alpha}\omega^\beta + T_{\beta\cdot}^{\ \alpha}\omega^{\bar{\beta}}$$

$$\psi = \psi' + G\omega + i(H_\alpha\omega^\alpha - H_{\bar{\alpha}}\omega^{\bar{\alpha}})$$

then by (3.5)

$$\begin{aligned} d\omega^\alpha &= \omega^\beta \wedge (\phi'^{\alpha}_{\beta\cdot} + D_{\beta\cdot}^{\alpha}\omega) + \omega \wedge (\phi'^\alpha + E^\alpha\omega + R_{\beta\cdot}^{\alpha}\omega^\beta + T_{\beta\cdot}^{\alpha}\omega^{\bar\beta}) \\ &= \omega^\beta \wedge \phi'^{\alpha}_{\beta\cdot} + \omega \wedge \phi'^\alpha \end{aligned}$$

$$\Rightarrow R_{\beta\cdot}^{\alpha} = D_{\beta\cdot}^{\alpha} \quad T_{\beta\cdot}^{\alpha} = 0$$

By (3.9)

$$\begin{aligned} d\phi &= i\omega_{\bar\beta} \wedge \phi^{\bar\beta} + i\phi_{\bar\beta} \wedge \omega^{\bar\beta} + \omega \wedge \psi \\ &= i\omega_{\bar\beta} \wedge \overline{(\phi'^\beta + E^\beta\omega + D_{\sigma\cdot}^{\beta}\omega^\sigma)} + i(\phi'^\gamma + E^\gamma\omega + D_{\sigma\cdot}^{\gamma}\omega^\sigma)g_{\gamma\bar\beta} \wedge \omega^{\bar\beta} \\ &\quad + \omega \wedge (\psi' + G\omega + i(H_\alpha\omega^\alpha - H_{\bar\alpha}\omega^{\bar\alpha})) \\ &= i\omega_{\bar\beta} \wedge \phi'^{\bar\beta} + i\phi'_{\bar\beta} \wedge \omega^{\bar\beta} + \omega \wedge \psi' \end{aligned}$$

$$\Rightarrow H_\alpha = \overline{E^\gamma g_{\gamma\bar\alpha}} = E_\alpha, \quad D_{\alpha\bar\beta} + D_{\bar\beta\alpha} = 0$$

□

Lemma 3.1.4 *The $D_{\beta\cdot}^{\alpha}$ can be uniquely determined by the conditions*

$$S_{\rho\bar\sigma} =: g^{\alpha\bar\beta}S_{\alpha\rho\bar\beta\bar\sigma} = 0 \tag{3.17}$$

Proof: Define $S = g^{\alpha\bar\beta}S_{\alpha\bar\beta}$ and $D = D_{\alpha\cdot}^{\alpha}$. Since $g^{\alpha\bar\beta}$ and $S_{\alpha\bar\beta}$ are hermitian and $D_{\alpha\bar\beta}$ is skew-hermitian, S is real and D is purely imaginary. Let

$$\begin{aligned} &\Pi'^{\gamma}_{\alpha\cdot} = d\phi'^{\gamma}_{\alpha\cdot} - \phi'^{\beta}_{\alpha\cdot} \wedge \phi'^{\gamma}_{\beta\cdot} \\ &\Phi'^{\gamma}_{\beta\cdot} \equiv \Pi'^{\gamma}_{\beta\cdot} - i\omega_\beta \wedge \phi'^\gamma + i\phi'_\beta \wedge \omega^\gamma + i\delta^\beta_\gamma(\phi'_\sigma \wedge \omega^\sigma) \quad mod\, \omega \end{aligned}$$

one has

$$\begin{aligned} &S_{\alpha\beta\cdot\bar\sigma}^{\ \ \gamma} = S'^{\ \ \gamma}_{\alpha\beta\cdot\bar\sigma} + i(D_{\alpha\cdot}^{\gamma}g_{\beta\bar\sigma} + D_{\beta\cdot}^{\gamma}g_{\alpha\bar\sigma} - \delta^\gamma_\beta D_{\bar\sigma\alpha} - \delta^\gamma_\alpha D_{\bar\sigma\beta}) \\ \Rightarrow\ &S_{\rho\bar\sigma} = S'_{\rho\bar\sigma} + i(g_{\rho\bar\sigma}D + D_{\rho\bar\sigma} - (n+1)D_{\bar\sigma\rho}) \end{aligned}$$

Therefore

$$S'_{\rho\bar\sigma} = 0 \ \Rightarrow\ -iS_{\rho\bar\sigma} = g_{\rho\bar\sigma}D + (n+2)D_{\rho\bar\sigma} \tag{3.18}$$

$$\Rightarrow\ 2(n+1)D = -iS \tag{3.19}$$

Substitute (3.19) into (3.18), one has

$$D_{\rho\bar{\sigma}} = \frac{-iS_{\rho\bar{\sigma}}}{n+2} + \frac{i}{2(n+1)(n+2)} S g_{\rho\bar{\sigma}} \tag{3.20}$$

Since the $D_{\rho\bar{\sigma}}$ given by (3.20) is unique and satisfies $D_{\rho\bar{\sigma}} + D_{\bar{\sigma}\rho} = 0$, the lemma is proved. □

By the condition (3.17) the forms $\phi_{\beta\cdot}^{\ \gamma}$ are completely determined and we wish to compute their exterior derivatives. By (3.15) and lemma 3.1.2, one has

$$\Pi_{\beta\cdot}^{\ \gamma} - i\omega_\beta \wedge \phi^\gamma + i\phi_\beta \wedge \omega^\gamma + i\delta_\beta^\gamma(\phi_\sigma \wedge \omega^\sigma) = S_{\beta\rho\cdot\bar{\sigma}}^{\ \ \gamma}\omega^\rho \wedge \omega^{\bar{\sigma}} + \lambda_{\beta\cdot}^{\ \gamma} \wedge \omega \tag{3.21}$$

where $\lambda_{\beta\cdot}^{\ \gamma}$ are one-forms. Substitute (3.21) into (3.7), one has

$$d\phi^\alpha - \phi \wedge \phi^\alpha - \phi^\beta \wedge \phi_{\beta\cdot}^{\ \alpha} - \lambda_{\beta\cdot}^{\ \alpha} \wedge \omega^\beta = \mu^\alpha \wedge \omega \tag{3.22}$$

where μ^α are one-forms. By (3.9), (3.14) and (3.21), one has

$$(\lambda_{\beta\bar{\alpha}} + \lambda_{\bar{\alpha}\beta}) \wedge \omega = g_{\alpha\bar{\beta}}\omega \wedge \psi$$

$$\Rightarrow \lambda_{\beta\bar{\alpha}} + \lambda_{\bar{\alpha}\beta} + g_{\beta\bar{\alpha}}\psi \equiv 0 \quad mod\ \omega \tag{3.23}$$

By (3.5), (3.8) and (3.22), one has

$$d\omega_\alpha = d(g_{\alpha\bar{\beta}}\omega^{\bar{\beta}}) = -\omega^{\bar{\beta}} \wedge \phi_{\alpha\bar{\beta}} + \omega_\alpha \wedge \phi + \omega \wedge \phi_\alpha \tag{3.24}$$

$$d\phi_\alpha = d(g_{\alpha\bar{\beta}}\phi^{\bar{\beta}}) = \phi_{\alpha\bar{\beta}} \wedge \phi^{\bar{\beta}} + \lambda_{\gamma\bar{\alpha}} \wedge \omega^{\bar{\gamma}} + \mu_\alpha \wedge \omega \tag{3.25}$$

By differentiating (3.21) and consider only terms involving $\omega^\rho \wedge \omega^{\bar{\sigma}}$ ignoring those in ω, one has

$$\begin{aligned} & dS_{\beta\rho\cdot\bar{\sigma}}^{\ \ \gamma} - S_{\tau\rho\cdot\bar{\sigma}}^{\ \ \gamma}\phi_{\beta\cdot}^{\ \tau} - S_{\beta\tau\cdot\bar{\sigma}}^{\ \ \gamma}\phi_{\rho\cdot}^{\ \tau} + S_{\beta\rho\cdot\bar{\sigma}}^{\ \ \tau}\phi_{\tau\cdot}^{\ \gamma} - S_{\beta\rho\cdot\bar{\tau}}^{\ \ \gamma}\phi_{\bar{\sigma}\cdot}^{\ \bar{\tau}} \\ \equiv\ & i(\lambda_{\beta\cdot}^{\ \gamma}g_{\rho\bar{\sigma}} + \lambda_{\rho\cdot}^{\ \gamma}g_{\beta\bar{\sigma}} - \delta_\beta^\gamma\lambda_{\bar{\sigma}\rho} - \delta_\rho^\gamma\lambda_{\bar{\sigma}\beta}) \quad mod\ \omega, \omega^\alpha, \omega^{\bar{\beta}} \end{aligned}$$

By contraction

$$dS_{\rho\bar{\sigma}} - S_{\tau\bar{\sigma}}\phi_{\rho\cdot}^{\ \tau} - S_{\rho\bar{\tau}}\phi_{\bar{\sigma}\cdot}^{\ \bar{\tau}} \equiv i(g_{\rho\bar{\sigma}}\lambda_{\beta\cdot}^{\ \beta} + \lambda_{\rho\bar{\sigma}} - (n+1)\lambda_{\bar{\sigma}\rho}) \quad mod\ \omega, \omega^\alpha, \omega^{\bar{\beta}}$$

When $S_{\rho\bar{\sigma}} = 0$, $g_{\rho\bar{\sigma}}\lambda_{\beta\cdot}^{\ \beta} + \lambda_{\rho\bar{\sigma}} - (n+1)\lambda_{\bar{\sigma}\rho} \equiv 0 \quad mod\ \omega, \omega^\alpha, \omega^{\bar{\beta}}$. By (3.23)

$$\Rightarrow \ \lambda_{\rho\bar{\sigma}} \equiv -\frac{1}{2}g_{\rho\bar{\sigma}}\psi \quad mod\ \omega, \omega^\alpha, \omega^{\bar{\beta}}$$
$$\Rightarrow \ \lambda_{\rho\cdot}^{\ \sigma} \equiv -\frac{1}{2}\delta_\rho^\sigma\psi \quad mod\ \omega, \omega^\alpha, \omega^{\bar{\beta}}$$

Therefore one can write

$$\lambda_{\rho\cdot}^{\ \sigma} = -\frac{1}{2}\delta_\rho^\sigma\psi + V_{\rho\cdot\beta}^{\ \sigma}\omega^\beta + W_{\rho\cdot\bar{\beta}}^{\ \bar{\sigma}}\omega^{\bar{\beta}} + a_{\rho\cdot}^{\ \sigma}\omega$$
$$\text{or} \quad \lambda_{\rho\bar{\sigma}} = -\frac{1}{2}g_{\rho\bar{\sigma}}\psi + V_{\rho\bar{\sigma}\beta}\omega^\beta + W_{\rho\bar{\sigma}\bar{\beta}}\omega^{\bar{\beta}} + a_{\rho\bar{\sigma}}\omega$$

Substitute into (3.23), one has

$$V_{\rho\bar{\sigma}\beta} + W_{\bar{\sigma}\rho\beta} = 0$$

Hence (3.21) can be written

$$\begin{aligned}\Phi_{\beta\cdot}^{\ \gamma} &= d\phi_{\beta\cdot}^{\ \gamma} - \phi_{\beta\cdot}^{\ \sigma} \wedge \phi_{\sigma\cdot}^{\ \gamma} - i\omega_\beta \wedge \phi^\gamma + i\phi_\beta \wedge \omega^\gamma + i\delta_\beta^\gamma(\phi_\sigma \wedge \omega^\sigma) + \frac{1}{2}\delta_\beta^\gamma\psi \wedge \omega \\ &= S_{\beta\rho\cdot\bar{\sigma}}^{\ \ \gamma}\omega^\rho \wedge \omega^{\bar{\sigma}} + V_{\beta\cdot\rho}^{\ \gamma}\omega^\rho \wedge \omega + W_{\beta\cdot\bar{\sigma}}^{\ \gamma}\omega^{\bar{\sigma}} \wedge \omega\end{aligned}$$

$$= S_{\beta\rho\cdot\bar{\sigma}}^{\ \ \gamma}\omega^\rho \wedge \omega^\sigma + V_{\beta\cdot\rho}^{\ \gamma}\omega^\rho \wedge \omega - V_{\cdot\beta\bar{\sigma}}^{\gamma}\omega^{\bar{\sigma}} \wedge \omega \tag{3.26}$$

Subsititute into (3.7) one has

$$\Phi^\alpha = d\phi^\alpha - \phi \wedge \phi^\alpha - \phi^\beta \wedge \phi_{\beta\cdot}^{\ \alpha} + \frac{1}{2}\psi \wedge \omega^\alpha = -V_{\beta\cdot\gamma}^{\ \alpha}\omega^\beta \wedge \omega^\gamma + V_{\cdot\beta\bar{\sigma}}^{\alpha}\omega^\beta \wedge \omega^{\bar{\sigma}} + \nu^\alpha \wedge \omega \tag{3.27}$$

where ν^α are one-forms.

Lemma 3.1.5 *With (3.8) and (3.17) fulfilled as in lemmas 3.1.1 and 3.1.2 there is unique set of ϕ^α satisfying*

$$V_{\beta\cdot\rho}^{\ \rho} = 0 \tag{3.28}$$

Proof: The effect of the transformation in lemma 3.1.3 with $D_{\beta\cdot}^{\ \alpha} = 0$ on $V_{\beta\cdot\rho}^{\ \gamma}$ is given by:

$$V_{\beta\cdot\rho}^{\ \gamma} = V_{\beta\cdot\rho}^{'\ \gamma} - i(\delta_\rho^\gamma E_\beta + \frac{1}{2}\delta_\beta^\gamma E_\rho)$$
$$\Rightarrow\quad V_{\beta\cdot\rho}^{\ \rho} = V_{\beta\cdot\rho}^{'\ \rho} - i(n+\frac{1}{2})E_\beta$$

For $V_{\beta\cdot\rho}^{'\ \rho} = 0$, we have $E_\beta = \frac{iV_{\beta\cdot\rho}^{\ \rho}}{n+\frac{1}{2}}$ $\square$

An expression for $d\psi$ can be obtained by differentiating (3.9) and using (3.5), (3.24) and (3.27):

$$\omega \wedge (-d\psi + \phi\wedge\psi + 2i\phi^\beta \wedge \phi_\beta - i\omega^\beta\wedge\nu_\beta - i\nu^\beta\wedge\omega_\beta) = 0$$

$$\Rightarrow \Psi =: d\psi - \phi\wedge\psi - 2i\phi^\beta\wedge\phi_\beta = -i\omega^\beta\wedge\nu_\beta - i\nu^\beta\wedge\omega_\beta + \rho\wedge\omega \tag{3.29}$$

where ρ is a one-form. Hence by differentiating (3.27) and considering only terms in $\omega^\rho\wedge\omega^{\bar\sigma}$, one has

$$dV_{\cdot\rho\bar\sigma}^{\alpha} - V_{\cdot\beta\bar\sigma}^{\alpha}\phi_{\rho\cdot}^{\ \beta} + V_{\cdot\rho\bar\sigma}^{\beta}\phi_{\beta\cdot}^{\ \alpha} - V_{\cdot\rho\bar\tau}^{\alpha}\phi_{\bar\sigma\cdot}^{\ \bar\tau} - V_{\cdot\rho\bar\sigma}^{\alpha}\phi = S_{\beta\rho\cdot\bar\sigma}^{\ \ \alpha}\phi^\beta + ig_{\rho\bar\sigma}\nu^\alpha + \frac{i}{2}\delta_\rho^\alpha\nu_{\bar\sigma} \quad mod\ \omega,\omega^\alpha,\omega^{\bar\beta} \tag{3.30}$$

$V_{\beta\cdot\rho}^{\ \rho} = 0$ is equivalent to $V_{\cdot\rho\bar\sigma}^{\alpha}g^{\rho\bar\sigma} = 0$. Its differentiation gives, by (3.8) and (3.30)

$$\nu^\gamma \equiv 0 \quad mod\ \omega,\omega^\alpha,\omega^{\bar\beta}$$

Therefore we can write

$$\nu^\gamma \equiv P_{\alpha\cdot}^{\ \gamma}\omega^\alpha + Q_{\bar\beta\cdot}^{\ \gamma}\omega^{\bar\beta} \quad mod\ \omega \tag{3.31}$$

Substitute into (3.27)

$$\Phi^\alpha = -V_{\beta\cdot\gamma}^{\ \alpha}\omega^\beta\wedge\omega^\gamma + V_{\cdot\beta\bar\sigma}^{\alpha}\omega^\beta\wedge\omega^{\bar\sigma} + P_{\beta\cdot}^{\ \alpha}\omega^\beta\wedge\omega + Q_{\bar\beta\cdot}^{\ \alpha}\omega^{\bar\beta}\wedge\omega \tag{3.32}$$

For future use we write down the formula

$$\Phi_\alpha = d\phi_\alpha - \phi_{\alpha\bar\beta}\wedge\phi^{\bar\beta} + \frac{1}{2}\psi\wedge\omega_\alpha$$

$$= -V_{\bar{\beta}\alpha\bar{\gamma}}\omega^{\bar{\beta}} \wedge \omega^{\bar{\gamma}} - V_{\alpha\bar{\gamma}\beta}\omega^{\beta} \wedge \omega^{\bar{\gamma}} + Q_{\beta\alpha}\omega^{\beta} \wedge \omega + P_{\bar{\beta}\alpha}\omega^{\bar{\beta}} \wedge \omega \tag{3.33}$$

Substitute (3.31) into (3.29) one has

$$\Psi = i(Q_{\alpha\beta}\omega^{\alpha} \wedge \omega^{\beta} - Q_{\bar{\alpha}\bar{\beta}}\omega^{\bar{\alpha}} \wedge \omega^{\bar{\beta}}) - i\tilde{P}_{\rho\bar{\sigma}}\omega^{\rho} \wedge \omega^{\bar{\sigma}} + \rho \wedge \omega \tag{3.34}$$

where

$$\tilde{P}_{\alpha\bar{\beta}} = P_{\alpha\bar{\beta}} + P_{\bar{\beta}\alpha} = \tilde{P}_{\bar{\beta}\alpha} \tag{3.35}$$

Lemma 3.1.6 *The real form ψ is completely determined by the condition*

$$\tilde{P}_{\alpha\cdot}^{\ \alpha} = 0 \tag{3.36}$$

Proof: ψ still undergo the transformation

$$\psi = \psi' + G\omega$$

Denoting the new coefficients by dashes, from (3.27) and (3.32)

$$\begin{aligned} P'^{\ \alpha}_{\beta\cdot} &= P_{\beta\cdot}^{\ \alpha} + \frac{1}{2}\delta^{\alpha}_{\beta}G \\ \Rightarrow \quad P'^{\ \alpha}_{\alpha\cdot} &= P_{\alpha\cdot}^{\ \alpha} + \frac{n}{2}G \end{aligned}$$

Besides, from (3.35) one has

$$\begin{aligned} \tilde{P}_{\alpha\cdot}^{\ \alpha} &= 2Re(P_{\alpha\cdot}^{\ \alpha}) \\ \Rightarrow \quad \tilde{P}'^{\ \alpha}_{\alpha\cdot} &= \tilde{P}_{\alpha\cdot}^{\ \alpha} + nG \end{aligned}$$

For $\tilde{P}'^{\ \alpha}_{\alpha\cdot} = 0$, one has $G = \frac{-\tilde{P}_{\alpha\cdot}^{\ \alpha}}{n}$. □

By differentiating (3.34) and using (3.29), considering only terms in $\omega^{\rho} \wedge \omega^{\bar{\sigma}}$, one has

$$d\tilde{P}_{\rho\bar{\sigma}} - \tilde{P}_{\tau\bar{\sigma}}\phi_{\rho\cdot}^{\ \tau} - \tilde{P}_{\rho\bar{\tau}}\phi_{\bar{\sigma}\cdot}^{\ \bar{\tau}} - \tilde{P}_{\rho\bar{\sigma}}\phi \equiv 2V_{\cdot\rho\bar{\sigma}}^{\beta}\phi_{\beta} + 2V_{\beta\bar{\sigma}\rho}\phi^{\beta} - g_{\rho\bar{\sigma}}\rho \quad mod\ \omega, \omega^{\alpha}, \omega^{\bar{\beta}}$$

By (3.8), (3.28) and (3.36), one has

$$\rho \equiv 0 \quad mod\ \omega, \omega^{\alpha}, \omega^{\bar{\beta}}$$

Hence (3.34) can be written

$$\Psi = i(Q_{\alpha\beta}\omega^{\alpha} \wedge \omega^{\beta} - Q_{\bar{\alpha}\bar{\beta}}\omega^{\bar{\alpha}} \wedge \omega^{\bar{\beta}}) - i\tilde{P}_{\rho\bar{\sigma}}\omega^{\rho} \wedge \omega^{\bar{\sigma}} + (R_{\alpha}\omega^{\alpha} + R_{\bar{\alpha}}\omega^{\bar{\alpha}}) \wedge \omega \quad (3.37)$$

The above derivation can be summarized in the following theorem:

Theorem 3.1.7 *Let the manifold M of dimension 2n+1 be provided with an integrable nondegenerate G-structure. Then the real line bundle E over M has a G_1-structure, in whose associated principal G_1-bundle Y there is a completely determined set of 1-forms $\omega, \omega^{\alpha}, \phi, \phi_{\beta\cdot}^{\alpha}, \phi^{\alpha}, \psi$ of which ω, ϕ, ψ are real, which satisfy the equations (3.3), (3.5), (3.8), (3.9), (3.17), (3.26), (3.27), (3.28), (3.29), (3.32), (3.36) and (3.37). The forms*

$$\omega, \omega^{\alpha}, \omega^{\bar{\alpha}}, \phi, \phi_{\alpha\cdot}^{\beta}, \phi^{\alpha}, \phi^{\bar{\alpha}}, \psi \quad (3.38)$$

are linearly independent. In particular, suppose that the G-structure arises from a real analytic real hypersurface M in $\mathbb{C}^{n+1}$. Suppose there is a second real analytic hypersurface M' in $\mathbb{C}'^{n+1}$ whose corresponding concepts are denoted by dashes. Then there is locally a biholomorphic transformation of $\mathbb{C}^{n+1}$ to $\mathbb{C}'^{n+1}$ which maps M to M' if and only if there is a real analytic diffeomorphism of Y to Y' under which the forms in (3.38) are respectively equal to the forms with dashes.

Proof: " $\Rightarrow$ " let $f : M \to M'$ be a local CR map. Let $(\theta, \theta^{\alpha})$ and $(\theta', \theta'^{\alpha})$ be the CR structures on M and M' respectively. Identifying M and M' on the same $\mathbb{C}^{n+1}$, one has

$$\begin{aligned} \theta' &= u\theta \\ \theta'^{\alpha} &= v^{\alpha}\theta + u_{\beta}^{\alpha}\theta^{\beta} \end{aligned}$$

and hence

$$\begin{aligned} \omega' &= \omega \\ \omega'^{\alpha} &= \omega^{\alpha} \end{aligned}$$

Then the remaining forms will be the same according to the above derivation. " $\Leftarrow$ " let (z^α, ω) and (z'^α, ω') be local coordinates on $\mathbb{C}^{n+1}$ and $\mathbb{C}'^{n+1}$ respectively. let $g : Y \to Y'$ be a real analytic diffeomorphism of Y to Y' under which the forms in (3.38) are respectively equal to the forms with dashes. ω, ω^α are linear combinations of dz^α, dw and are linearly independent over $\mathbb{C}$. Since

$$\begin{aligned} \omega' &= \omega \\ \omega'^\alpha &= \omega^\alpha \end{aligned}$$

g has the property that $dz'^\alpha, d\omega'$ are linear combinations of dz^β, dw. Since

$$dz'^\alpha = \frac{\partial z'^\alpha}{\partial z^\beta} dz^\beta + \frac{\partial z'^\alpha}{\partial z^{\bar\beta}} dz^{\bar\beta} + \frac{\partial z'^\alpha}{\partial w} dw + \frac{\partial z'^\alpha}{\partial \bar w} d\bar w$$

Then $\frac{\partial z'^\alpha}{\partial z^{\bar\beta}} = \frac{\partial z'^\alpha}{\partial \bar w} = 0 \Rightarrow (z'^\alpha, \omega')$ are holomorphic functions of z and w. $\square$

3.2 Geometric interpretation of the solution

In this section, we first apply the results in §3.1 to the real hyperquadrics

$$Q = \{(z, w) \mid v = g_{\alpha\bar{\beta}} z^\alpha z^{\bar{\beta}}\}$$

and then consider the general case. The notations in §1.3 will be used. Let Z_A be a Q-frame with Z_0 lies in Q where $Z_0 = tY, Y = (1, z^1, ..., z^n, \omega)$. Then

$$\frac{i}{2}\pi_0^{n+1} = (dZ_0, Z_0) = |t|^2(dY, Y) = |t|^2(\frac{i}{2}dw + g_{\alpha\bar{\beta}} z^{\bar{\beta}} dz^\alpha)$$

On the other hand,

$$\omega = u\theta = u(\frac{1}{2}dw - i g_{\alpha\bar{\beta}} z^{\bar{\beta}} dz^\alpha)$$

By setting, $u = |t|^2$, one has

$$\omega = \frac{1}{2}\pi_0^{n+1}$$

Comparing (3.3) with the structure equation (1.9) for $d\pi_0^{n+1}$, one can put

$$\omega^\alpha = \pi_0^\alpha \quad \phi = -\pi_0^0 + \pi_{n+1}^{n+1} = -\pi_0^0 - \bar{\pi}_0^0 \tag{3.39}$$

By putting

$$\phi^\alpha = 2\pi_{n+1}^\alpha \quad \phi_{\alpha\cdot}^{\ \beta} = \pi_{\alpha\cdot}^{\ \beta} - \delta_\alpha^\beta \pi_0^0 \quad \psi = -4\pi_{n+1}^0 \tag{3.40}$$

the equations (1.9) are identical to the equations in Theorem 3.1.7 with

$$S_{\alpha\beta\bar{\rho}\bar{\sigma}} = V_{\alpha\bar{\beta}\rho} = P_{\alpha\bar{\beta}} = Q_{\alpha\bar{\beta}} = R_\alpha = 0$$

Under the change of Q-frame,

$$\omega^* = -i(dZ_0^*, Z_0^*) = -i|t|^2(dZ_0, Z_0) = |t|^2\omega$$

let

$$H_1 = \left\{ \begin{pmatrix} t & 0 & 0 \\ t_\alpha & t_\alpha^\beta & 0 \\ \tau & \tau^\beta & \bar{t}^{-1} \end{pmatrix} \in H : \ |t| = 1 \right\}$$

Then the form ω is invariant under H_1. Since $dZ_A = \pi_A^B Z_B$, one has

$$\pi_0^0 = 2i(dZ_0, Z_{n+1}) \quad \pi_0^\alpha = g^{\alpha\bar{\beta}}(dZ_0, Z_\beta)$$

By (3.39), under a change of Q-frame by H_1, one has

$$\begin{pmatrix} \omega^* \\ \omega^{*\alpha} \\ \omega^{*\bar{\alpha}} \\ \phi^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ itt^\alpha & tt^\alpha_{\cdot\beta} & 0 & 0 \\ -it^{-1}t^{\bar{\alpha}} & 0 & t^{-1}t^{\bar{\alpha}}_{\cdot\bar{\beta}} & 0 \\ Re(\tau t^{-1}) & -2it\tau_\alpha & 2it^{-1}\tau_{\bar{\alpha}} & 1 \end{pmatrix} \begin{pmatrix} \omega \\ \omega^\beta \\ \omega^{\bar{\beta}} \\ \phi \end{pmatrix} \tag{3.41}$$

The coefficient matrix in (3.41) belongs to G_1. The mapping $H_1 \to G_1$ is a homomorphism. If $K = \{M \in K \mid M = \varepsilon I, \varepsilon^{n+2} = 1\}$, then one has the isomorphism

$$H_1/K \cong G_1$$

Since $\Gamma =: SU(p+1, q+1)/K \supset H_1/K \cong G_1$, in what follows G_1 will be considered a subgroup of $SU(p+1, q+1)/K$.

Introduce the matrix notation

$$(h) = \mathcal{H} = (h_{A\bar{B}})$$

where $\mathcal{H}$ is defined in (1.2). Then the lie algebra Lu of $SU(p+1, q+1)$ is given by

$$\{(l) = (l_{A\cdot}^{B}), 0 \le A, B \le n+1 \mid (l)(h) + (h)(l)^* = 0, Tr(l) = 0\}$$

And the lie algebra of H_1 is given by

$$\{(l) \in Lu \mid l_0^\alpha = l_0^{n+1} = Re(l_0^0) = 0\}$$

With this notation the matrix

$$(\pi) = (\pi_A^B)$$

is an Lu-valued one-form on $SU(p+1, q+1)$. The equations (1.10) can be written

$$d(\pi) = (\pi) \wedge (\pi)$$

Let

$$(Z)^T = (Z_0, Z_1, ..., Z_{n+1})$$

Then equations (1.8) can be written

$$d(Z) = (\pi)(Z)$$

and the equations for change of Q-frames become

$$(Z^*) = (t)(Z)$$

If (π^*) is defined by

$$d(Z^*) = (\pi^*)(Z^*)$$

then one has

$$(\pi^*) = (t)(\pi)(t)^{-1} =: ad(t)(\pi)$$

In general, let Y be a principal G_1-bundle over a manifold E. Let Γ be a linear group which contains G_1 as a subgroup. In our case,

$$\Gamma = SU(p+1, q+1)/K \supset H_1/K \cong G_1$$

Let γ be the Lie algebra of Γ realized as a Lie algebra of matrices. Then one has the adjoint transformation

$$\begin{array}{ccc} G_1 \times \Gamma & \longmapsto & \Gamma \\ (t) \quad (s) & & (t)(s)(t)^{-1} \end{array}$$

then G_1 acts on γ by the adjoint transformation

$$\begin{array}{ccc} G_1 \times \gamma & \longmapsto & \gamma \\ (t) \quad (l) & & (t)(l)(t)^{-1} \end{array}$$

Definition 3.2.1 *A Γ-connection in the bundle Y is a γ-valued 1-form (π), the connection form, such that under a change of frame by the group G_1, (π) transforms according to the formula*

$$(\pi^*) = ad(t)(\pi),\ (t) \in G_1$$

Definition 3.2.2 *The curvature form of a Γ-connection (Π) is defined by*

$$(\Pi) = d(\pi) - (\pi) \wedge (\pi)$$

Remark

The adjoint transformation of G_1 on γ leaves the Lie algebra g_1 of G_1 invariant and induces an action on the quotient space γ/g_1. The projection of the curvature form on γ/g_1 is called the torsion form.

In order to provide a geometric interpretation to the results in §3.1, one follows the case of the real hyperquadrics and rewrites the equations in Theorem 3.1.7 (the $g_{\alpha\bar\beta}$ are now supposed to be constants) by the following transformation:

$$\begin{aligned}
&\pi_0^{n+1} = 2\omega & &-(n+2)\pi_0^0 = \phi_{\alpha\cdot}^{\ \alpha} + \phi \\
&\pi_0^{\alpha} = \omega^{\alpha} & &\pi_{\alpha}^{n+1} = 2i\omega_{\alpha} \\
&\pi_{n+1}^{\alpha} = \frac{1}{2}\phi^{\alpha} & &\pi_{\alpha}^{0} = -i\phi_{\alpha} \\
&\pi_{\alpha\cdot}^{\ \beta} = \phi_{\alpha\cdot}^{\ \beta} + \delta_{\alpha}^{\beta}\pi_0^0 & &\pi_{n+1}^{n+1} = -\bar\pi_0^0 \\
&\pi_{n+1}^{0} = -\frac{1}{4}\psi & &
\end{aligned}$$

Then π_A^B are 1-forms in Y and the matrix $(\pi) = (\pi_A^B)$ is Lu-valued, i.e.,

$$(\pi)(h) + (h)(\bar\pi)^T = 0$$

and the equations in Theorem 3.1.7 can be written

$$d(\pi) = (\pi) \wedge (\pi) + (\Pi)$$

where

$$(\Pi) = \begin{pmatrix} \Pi_0^0 & 0 & 0 \\ \Pi_\alpha^0 & \Pi_\alpha^\beta & 0 \\ \Pi_{n+1}^0 & \Pi_{n+1}^\beta & -\bar\Pi_0^0 \end{pmatrix}$$

and

$$\begin{aligned}
&(n+2)\Pi_0^0 = -\Phi_{\alpha\cdot}^{\ \alpha} & &\Pi_{n+1}^0 = -\frac{1}{4}\Psi \\
&\Pi_\alpha^0 = -i\Phi_\alpha & &\Pi_{n+1}^\beta = \frac{1}{2}\Phi^\beta \\
&\Pi_\alpha^\beta = \Phi_{\alpha\cdot}^{\ \beta} - \frac{1}{n+2}\delta_\alpha^\beta\Phi_{\gamma\cdot}^{\ \gamma} & &
\end{aligned}$$

where $\Phi_{\alpha\cdot}^{\ \beta}$, Φ^α and Φ are exterior 2-forms in ω, ω^α and $\omega^{\bar\beta}$ defined in §3.1. For any such forms

$$\Theta \equiv a_{\alpha\bar\beta}\omega^\alpha \wedge \omega^{\bar\beta} + \text{terms quadratic in } \omega^\rho \text{ or } \omega^{\bar\sigma} \quad mod\ \omega$$

define

$$\mathrm{Tr}\Theta = g^{\alpha\bar\beta}a_{\alpha\bar\beta}$$

Then the equations (3.17), (3.28) and (3.36) can be written

$$Tr\Pi_\alpha^\beta = 0, \quad Tr\Pi_0^0 = 0$$
$$Tr\Pi_\beta^0 = Tr\Pi_{n+1}^\alpha = 0$$
$$Tr\Pi_{n+1}^0 = 0$$

or in the matrix equation

$$Tr(\Pi) = 0 \tag{3.42}$$

Under the adjoint transformation of H_1,

$$(\pi) \to ad(t)(\pi),$$
$$(\Pi) \to ad(t)(\Pi),$$

the condition (3.42) remains invariant. One submits ω, ω^α, $\omega^{\bar\beta}$, ϕ to the linear transformation with the coefficient matrix of (3.4) and denotes the new quantities by the same symbols with asterisks. Since (π) is uniquely determined by (3.42) according to theorem 3.1.7 and since these conditions are invariant under the adjoint transformation by H_1, one has

$$(\pi^*) = ad(t)(\pi), t \in G_1$$

Therefore (π) satisfies the conditions of a connection form and one has the theorem:

Theorem 3.2.3 *Given a nondegenerate integrable G-structure on a manifold M of dimension 2n+1. Consider the principal bundle Y over E with the group*

$G_1 \subset SU(p+1,q+1)/K$. *There is in* Y *a uniquely defined connection with the group* $SU(p+1,q+1)$, *which is characterized by the vanishing of the torsion form and the condition (3.42).*

We end this chapter by giving the definition of chains. Consider a curve γ which is everywhere transversal to the complex tangent hyperplane. Its tangent line can be defined by $\omega^\alpha = 0$. By (3.5) restricted to γ, one has

$$\phi^\alpha = b^\alpha \omega$$

Definition 3.2.4 *A curve* γ *is called a chain if* $b^\alpha = 0$. *The chains are therefore defined by the differential system*

$$\omega^\alpha = \phi^\alpha = 0$$

Chapter 4

Chains

4.1 Identification of the two definitions of chains

We gave two definitions of chains in the previous chapters. In this chapter, we study the result that the two invariant families of curves are actually the same and will discuss some of their properties. For simplicity we restrict ourselves to nondegenerate real hypersurfaces M $\in \mathbb{C}^2$ and assume $g_{1\bar{1}} = 1$ from now on.

In $\mathbb{C}^2$, Y is a 8 dimensional principal G_1 bundle and there are 8 linearly independent forms:

$$\omega \quad \omega^1 \quad \omega^{\bar{1}} \quad \phi \quad \phi_1^1 \quad \phi^1 \quad \phi^{\bar{1}} \quad \psi$$

satisfying

$$\begin{aligned}
d\omega &= i\omega^1 \wedge \omega^{\bar{1}} + \omega \wedge (\phi_1^1 + \bar{\phi_1^1}) \\
d\omega^1 &= \omega^1 \wedge \phi_1^1 + \omega \wedge \phi^1 \\
d\phi^1 &= (\phi_1^1 + \bar{\phi_1^1}) \wedge \phi^1 + \phi^1 \wedge \phi_1^1 - \tfrac{1}{2}\psi \wedge \omega^1 + Q\omega^{\bar{1}} \wedge \omega \\
d\psi &= (\phi_1^1 + \bar{\phi_1^1}) \wedge \psi + 2i\phi^1 \wedge \phi^{\bar{1}} + (R\omega^1 + \bar{R}\omega^{\bar{1}}) \wedge \omega \\
d\phi_1^1 &= i\omega^{\bar{1}} \wedge \phi^1 - 2i\phi^{\bar{1}} \wedge \omega^1 - \tfrac{1}{2}\psi \wedge \omega
\end{aligned} \tag{4.1}$$

By the isomorphism

$$
\begin{aligned}
\omega &= \Omega \\
\omega^1 &= \Omega_1 \\
\phi_1^1 &= -\Omega_2 \\
\phi^1 &= -\Omega_3 \\
\psi &= 2\Omega_4 \\
Q &= -R \\
R &= 2S
\end{aligned}
$$

(4.1) becomes

$$
\begin{aligned}
d\Omega &= i\Omega_1 \wedge \bar{\Omega_1} - \Omega \wedge (\Omega_2 + \bar{\Omega_2}) \\
d\Omega_1 &= -\Omega_1 \wedge \Omega_2 - \Omega \wedge \Omega_3 \\
d\Omega_2 &= 2i\Omega_1 \wedge \bar{\Omega_3} + i\bar{\Omega_1} \wedge \Omega_3 - \Omega \wedge \Omega_4 \\
d\Omega_3 &= -\Omega_1 \wedge \Omega_4 - \bar{\Omega_2} \wedge \Omega_3 - R\Omega \wedge \bar{\Omega_1} \\
d\Omega_4 &= i\Omega_3 \wedge \bar{\Omega_3} - (\Omega_2 + \bar{\Omega_2}) \wedge \Omega_4 + (S\Omega_1 + \bar{S}\bar{\Omega_1}) \wedge \Omega
\end{aligned}
\tag{4.2}
$$

According to definition (3.2.4), $\omega^\alpha = \phi^\alpha = 0$ gives $\Omega_1 = \Omega_3 = 0$.

Cartan constructed a complete set of invariant forms starting with a real 1 forms θ and $\theta^1 = dz$. θ is normalized by the equation:

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}} + b\theta \wedge \theta^1 + \bar{b}\theta \wedge \theta^{\bar{1}}$$

By introducing auxiliary variables $\lambda \in \mathbb{C} \setminus \{0\}, \mu \in \mathbb{C}, \rho \in \mathbb{R}$, Cartan defined the one-forms

$$
\begin{aligned}
\Omega &= |\lambda|^2\theta \\
\Omega_1 &= \lambda(\theta^1 + \mu\theta) \\
\Omega_2 &= \tfrac{d\lambda}{\lambda} + A\theta^1 + B\theta^{\bar{1}} + C\theta \\
\Omega_3 &= \tfrac{1}{\lambda}(d\mu + D\theta^1 + E\theta^{\bar{1}} + F\theta) \\
\Omega_4 &= \tfrac{1}{|\lambda|^2}\left(d\rho + \tfrac{i}{2}(\mu d\bar{\mu} - \bar{\mu} d\mu) + H\theta^1 + \bar{H}\theta^{\bar{1}} + G\theta\right)
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
A &= -(b+2i\bar{\mu}) \\
B &= -i\mu \\
C &= \rho+\frac{ic}{4}-\frac{3i}{2}|\mu|^2, \quad c=b_{\bar{1}} \\
D &= \rho+\frac{ic}{4}+\frac{i}{2}|\mu|^2 \\
E &= -\mu(\bar{b}-i\mu) \\
F &= \frac{i\mu|\mu|^2}{2}+\mu\rho-\frac{ic\mu}{4}+\frac{c_{\bar{1}}-\bar{b}c+2i\bar{b}_o}{6} \\
G &= \frac{11}{48}c^2+\frac{|b|^2c+b\bar{l}+\bar{b}l-g}{6}+\frac{i}{6}(\mu l-\bar{\mu}\bar{l})-\frac{c|\mu|^2}{4}+\rho^2+\frac{|\mu|^4}{4} \\
& \quad l=c_1-bc-2ib_0 \quad g=c_{1\bar{1}}-i\frac{c_0}{2} \\
H &= \frac{|\mu|^2}{\mu}-i\bar{\mu}\rho-b\rho-\frac{1}{2}ib|\mu|^2+\frac{1}{4}c\bar{\mu}+\frac{1}{2}b_0-\frac{1}{12}il
\end{aligned}
$$

and the $b_o, b_1, b_{\bar{1}}$ are defined by the equation:

$$db = b_o\theta + b_1\theta^1 + b_{\bar{1}}\theta^{\bar{1}}$$

The exterior differentiations of the forms are the same as the equations given by (4.2). The variables (x, μ, λ, ρ) are local coordinates for the 8 dimensional bundle Y. $\Omega_1 = \Omega_3 = 0$ gives:

$$
\begin{aligned}
\theta^1 &= -\mu\theta \\
d\mu &= (i\mu|\mu|^2 - \bar{b}|\mu|^2 + \tfrac{1}{2}ic\mu - \tfrac{1}{6}\bar{l})\theta
\end{aligned}
\tag{4.4}
$$

Proposition 4.1.1 *Through each point $p \in M$ and tangent to a vector transversal to the complex tangent hyperplane H there passes exactly one chain.*

Proof: $\{\Omega_1, \Omega_3, \bar{\Omega_1}, \bar{\Omega_3}\}$ defines a closed differential ideal. By the Frobenius Theorem, locally Y is foliated by some 4 dimensional bundles N. From the expressions (4.3) for Ω_1 and Ω_3, N must contain the vectors $\{\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\bar{\lambda}}, \frac{\partial}{\partial\rho}\}$ in its tangent space at each of its points. So N must be of the form $\{x(t), \mu(t), \lambda, \rho \mid t \in \mathbb{R}\}$. Since $\theta(\frac{dx}{dt}) \neq 0$, $x(t)$ is a smooth curve in M and $\pi: Y \to M$ takes N onto this curve. At any point p of M the coordinates can be introduced so that $\theta = du$ and $\theta^1 = dz$. Let $\gamma = (z(t), u(t))$ be a curve in M. Then $\Omega_1 \mid_\gamma = 0$ gives:

$$\mu(t) = -\frac{z'(t)}{u'(t)}$$

so μ is the reciprocal of the usual complex slope. We have $\mu = 0$ for the u-axis and $|\mu| = \infty$ for directions in z-plane. Any unoriented direction at a point x_o not in the H-plane has a finite slope μ_o and so the unique integral manifold N through $(x_0, \mu_0, \lambda, \rho)$ projects onto a chain through x_o and having direction μ_o. The uniqueness of the chain follows from the uniqueness of N through $(x_0, \mu_0, \lambda, \rho)$. □

Proposition 4.1.2 *The chains on Q are the intersection of Q with complex lines.*

Proof: " $\Leftarrow$ " On the real hyperquadric $Q = \{(z_1, z_2) \mid Imz_2 = |z_1|^2\}$ one may take $\theta = i\partial r = \frac{1}{2}dz_2 - i\bar{z}_1 dz_1$ and $\theta^1 = dz_1$. So $d\theta = i\theta^1 \wedge \theta^{\bar{1}}$ and $b = c = l = 0$. By (4.4) one has

$$\theta^1 = -\mu\theta$$

$$d\mu = i\mu|\mu|^2\theta$$

On the complex line $\alpha z_1 + \beta z_2 = \gamma$, one has $\alpha dz_1 + \beta dz_2 = 0$ and so on the intersection of this line with Q, θ and θ^1 are related by

$$\alpha\theta^1 + \beta(2\theta + 2i\bar{z}_1\theta^1) = 0$$

If one set $\mu = \frac{2\beta}{\alpha + 2i\beta\bar{z}_1}$, one has

$$\theta^1 = -\mu\theta$$

Besides,
$$0 = (2i\beta d\bar{z}_1)\mu + (\alpha + 2i\beta\bar{z}_1)d\mu$$

$$\Rightarrow d\mu = \frac{2i\beta|\mu|^2}{\alpha + 2i\beta\bar{z}_1}\theta = i\mu|\mu|^2\theta$$

Thus the intersection of the line with Q is a chain.

" $\Rightarrow$ " A direction transversal to H determines both a unique chain and a unique complex line. $\square$

To find explicitly the chains on Q, one take

$$\theta = \frac{1}{2}(du + iz_1 d\bar{z}_1 - i\bar{z}_1 dz_1)$$
$$\theta^1 = dz_1$$

It is convenient to choose a time parameter such that along a given chain $(z(t), u(t))$

$$\theta = \frac{1}{|\mu|^2}$$

and so $\theta_1 = -\frac{1}{\bar{\mu}}$ and $d\mu = i\mu$

That is, one consider the system

$$\begin{aligned} u'(t) &= \tfrac{2}{|\mu(t)|^2} - iz_1(t)\overline{z_1'(t)} + i\overline{z_1(t)}z_1'(t) \\ z_1'(t) &= -\tfrac{1}{\bar{\mu}(t)} \\ \mu'(t) &= i\mu(t) \end{aligned} \tag{4.5}$$

ignoring the solution given by the u-axis. The unique solution to (4.5) with the initial value

$$u(0) = 0 \quad z_1(0) \quad \mu(0) = \nu$$

is:

$$z_1(t) = \frac{i}{\bar{\nu}}(e^{it} - 1) \quad u(t) = \frac{2sint}{|\nu|^2} \quad \mu(t) = \nu e^{it}$$

Theorem 4.1.3 *The chains given by definitions (2.2.11) and (3.2.4) are the same.*

Proof: Let M be in partial normal form

$$v = | z |^2 + \sum_{k\geq 2, l\geq 2} F_{kl}(u) z^k \bar{z}^l$$

$$= |z|^2 + F(z, \bar{z}, u) \tag{4.6}$$

Here one assume F be more general and is of order $\mathcal{O}|z|^K$ for some $K \geq 4$. In what follows we want to express b, c etc. in terms of the defininig function

$$r = \frac{1}{2i}(z_2 - \bar{z}_2) - v(z, \bar{z}, u)$$

One computes

$$\theta = i\partial r = \left(\frac{1}{2} + \frac{1}{2}v_u^2\right) du + \left(-\frac{i}{2} + \frac{1}{2}v_u\right) v_z dz + \left(\frac{i}{2} + \frac{1}{2}v_u\right) v_{\bar{z}} d\bar{z} \tag{4.7}$$

Subsititute (4.6) into (4.7) and use $F = \mathcal{O}|z|^K$, $K \geq 4$, one has

$$i\partial r = \left(\frac{1}{2} + \mathcal{O}|z|^{2K}\right) du + \left(-\frac{i}{2}\bar{z} - \frac{i}{2}A + \mathcal{O}|z|^{2K-1}\right) dz + \left(\frac{i}{2}z + \frac{i}{2}\bar{A} + \mathcal{O}|z|^{2K-1}\right) d\bar{z} \tag{4.8}$$

where $$A = F_z + i\bar{z}F_u$$

$$\Rightarrow d(i\partial r) = i(1 + B')dz \wedge d\bar{z} + i(\frac{1}{2}A_u + \mathcal{O}|z|^{2K-1})dz \wedge du - i(\frac{1}{2}\overline{A_u} + \mathcal{O}|z|^{2K-1})d\bar{z} \wedge du \tag{4.9}$$

where $$B' = \tfrac{1}{2}(A_{\bar{z}} + \bar{A}_z) + \mathcal{O}|z|^{2K-2}$$

By (4.7) $du = (2 + \mathcal{O}|z|^{2K})i\partial r + \mathcal{O}|z|dz + \mathcal{O}|z|d\bar{z}$, so (4.9) can be rewritten as

$$d(i\partial r) = i(1 + B)dz \wedge d\bar{z} + (iA_u + \mathcal{O}|z|^K)dz \wedge (i\partial r) + (-i\overline{A_u} + \mathcal{O}|z|^K)d\bar{z} \wedge (-i\partial r)$$

where $$B = \tfrac{1}{2}(A_{\bar{z}} + \bar{A}_z) + \mathcal{O}|z|^K$$

Let $f = \frac{1}{1+B}$ and take $\theta = if\partial r$ and $\theta^1 = dz$, then

$$d\theta = idz \wedge d\bar{z} + (iA_u + \mathcal{O}|z|^K)dz \wedge \theta + (-i\bar{A}_u + \mathcal{O}|z|^K)d\bar{z} \wedge \theta + \frac{df}{f}\theta$$

Define the coefficients a and β by the equation:

$$\theta = adu + \beta dz + \bar{\beta}d\bar{z}$$

Then it can be shown that for any function h with $dh = h_0\theta + h_1 dz + h_{\bar{1}} d\bar{z}$,

$$h_0 = a^{-1}h_u$$
$$h_1 = h_z - a^{-1}\beta h_u$$
$$h_{\bar{1}} = h_{\bar{z}} - a^{-1}\bar{\beta}h_u$$

Since $f = \frac{1}{1+B} = 1 - B + B^2 - \ldots$ and $B = \mathcal{O}|z|^{K-2}$, so $f = 1 + \mathcal{O}|z|^{K-2}$. By (4.8), one has

$$\theta = f(i\partial r) = \left(\frac{1}{2} + \mathcal{O}|z|^{K-2}\right) du + \left(-\frac{i}{2}\bar{z} + \mathcal{O}|z|^{K-1}\right) dz + \left(\frac{i}{2}z + \mathcal{O}|z|^{K-1}\right) d\bar{z}$$

and so

$$a^{-1} = 2 + \mathcal{O}|z|^{K-2}$$
$$a^{-1}\beta = -i\bar{z} + \mathcal{O}|z|^{K-1}$$
$$a^{-1}\bar{\beta} = iz + \mathcal{O}|z|^{K-1}$$

Thus

$$h_0 = (2 + \mathcal{O}|z|^{K-2})h_u$$
$$h_1 = h_z + (i\bar{z} + \mathcal{O}|z|^{K-1})h_u$$
$$h_{\bar{1}} = h_{\bar{z}} - (iz + \mathcal{O}|z|^{K-1})h_u$$

Hence

$$d\theta = idz \wedge d\bar{z} + b\theta \wedge dz + \bar{b}\theta \wedge d\bar{z} \tag{4.10}$$

$$\text{where} \quad b = -iA_u - f_z - Bf_z - i\bar{z}f_u + \mathcal{O}|z|^K \tag{4.11}$$

Further,

$$\begin{aligned} c &= b_{\bar{1}} = b_{\bar{z}} - izb_u + \mathcal{O}|z|^{K-1} \\ c_1 &= c_z + i\bar{z}c_u + \mathcal{O}|z|^{K-1} \\ c_{1\bar{1}} &= c_{1\bar{z}} - izc_1 + \mathcal{O}|z|^{K-1} \\ l &= c_1 - bc - 2ib_0 \end{aligned} \tag{4.12}$$

Since $B = \mathcal{O}|z|^{K-2}$ and $f = 1 - B + \mathcal{O}|B|^2$, (4.11) can be written $b = -iA_u + B_z + \mathcal{O}|z|^{K-1}$. Corresponding to this simplification, one has

$$
\begin{aligned}
b &= F_{zz\bar{z}} + \mathcal{O}|z|^{K-1} \\
c &= F_{zz\bar{z}\bar{z}} + \mathcal{O}|z|^{K-2} \\
c_1 &= F_{zzz\bar{z}\bar{z}} + \mathcal{O}|z|^{K-3} \\
c_{1\bar{1}} &= F_{zzz\bar{z}\bar{z}\bar{z}} + \mathcal{O}|z|^{K-4} \\
l &= F_{zzz\bar{z}\bar{z}} + \mathcal{O}|z|^{K-3}
\end{aligned}
\tag{4.13}
$$

Recall that we assume our hypersurface M is in partial normal form

$$
v = | z |^2 + \sum_{k\geq 2, l\geq 2} F_{kl}(u) z^k \bar{z}^l
$$

For M in normal form, $K = 4$. Then b, c, l in (4.12) become

$$
\begin{aligned}
b &= 4F_{22}\bar{z} + 12F_{32}|z|^2 + 6F_{23}\bar{z}^2 + (18F_{33} - 16F_{22}^2 + 2iF_{22}')z\bar{z}^2 + 8F_{24}\bar{z}^3 + 24F_{24}z^2\bar{z} + \mathcal{O}|z|^4 \\
c &= 4F_{22} + 12(F_{32}z + F_{23}\bar{z}) + 4(9F_{33} - 8F_{22}^2)|z|^2 + 24(F_{24}\bar{z}^2 + F_{42}z^2) + \mathcal{O}|z|^3 \\
l &= 4(3F_{32} + 12F_{42}z + (9F_{33} - 3iF_{22}' - 4F_{22}^2)\bar{z}) + \mathcal{O}|z|^2
\end{aligned}
\tag{4.14}
$$

And the chains on M satisfy the equations

$$
\begin{aligned}
dz &= -\mu\theta \\
d\mu &= (-2F_{23}(u) + \mathcal{O}|z| + \mathcal{O}|\mu|)\theta
\end{aligned}
\tag{4.15}
$$

where $\mathcal{O}|\mu|$ is small with respect to $|\mu|$.

Now let $p \in M$, $v \in TM_p$ be a direction transversal to H, γ be the unique chain through p in the direction v in the sense of definition (2.2.11), Γ be the unique chain through p in the direction v in the sense of definition (3.2.4), Φ be a local biholomorphism taking p to the origin, γ to the u-axis and M to a hypersurface in normal form. Then Γ is mapped to the unique curve in $\Phi(M)$ passing through the origin and tangent there to the u-axis. The curve $\Phi(\Gamma)$ is a chain in the sense of definition (3.2.4).

To show that $\gamma = \Gamma$ we need to show that u-axis is a chain in the sense of definition (3.2.4). But along the u-axis, $z \equiv \mu \equiv 0$, which is a solution of (4.15) as long as $F_{32}(u) \equiv 0$. $\square$

4.2 Chain-preserving maps

A CR diffeomorphism preserves chains. In this section we study Cheng's result [Ch] that the converse is essentially true. We start with the following lemma.

Lemma 4.2.1 *The CR structure $(M, \mathbb{L})$ and its conjugate CR structure $(M, \bar{\mathbb{L}})$ have the same chains.*

Proof: Let the CR structure of $(M, \mathbb{L})$ be given by $(\theta, \theta^1, \theta^{\bar{1}})$, normalized so that $d\theta = i\theta^1\theta^{\bar{1}} \quad mod\ \theta$. Let $\omega = -\theta$, $\omega^1 = \theta^{\bar{1}}$. Define, as in (4.3)

$$\begin{aligned}
W &= |l|^2\omega \\
W_1 &= l(\omega^1 + u\omega) \\
W_2 &= \tfrac{dl}{l} + \tilde{A}\omega^1 + \tilde{B}\omega^{\bar{1}} + \tilde{C}\omega \\
W_3 &= \tfrac{1}{l}(du + \tilde{D}\omega^1 + \tilde{E}\omega^{\bar{1}} + \tilde{F}\omega) \\
W_4 &= \tfrac{1}{|l|^2}(dr + \tilde{G}\omega + \tfrac{i}{2}udu - \tfrac{i}{2}\bar{u}d\bar{u} + \tilde{H}\omega^1 + \bar{\tilde{H}}\omega^{\bar{1}})
\end{aligned}$$

Then

$$\begin{aligned}
W \quad &= -\Omega \\
W_1 \quad &= \bar{\Omega}_1 \\
W_2 \quad &= \bar{\Omega}_2 \\
W_3 \quad &= -\bar{\Omega}_3 \\
W_4 \quad &= -\Omega_4
\end{aligned}$$

and $\lambda = \bar{l}$, $\mu = -\bar{u}$, $\rho = -r$. Since $\{W_1, \bar{W}_1, W_3, \bar{W}_3\}$ and $\{\Omega_1, \bar{\Omega}_1, \Omega_3, \bar{\Omega}_3\}$ generate the same differential ideal, the intergral submanifolds and thus the projections into M coincide. $\square$

Theorem 4.2.2 *Let $\phi : M \to \tilde{M}$ be a diffeomorphism between strictly pseudoconvex 3 dimensional CR structures. If ϕ maps the chains of M to the chains of $\tilde{M}$ then ϕ is either a CR diffeomorphism or a conjugate CR diffeomorphism.*

Proof: Since ϕ is a diffeomorphism, $\phi_*|_p$ is a linear isomorphism of T_pM to $T_{\phi(p)}\tilde{M}$. Since ϕ preserves chains, ϕ_* maps each direction transversal to H_p to a direction transversal to $H_{\phi(p)}$. It follows that ϕ_* maps H_p to $H_{\phi(p)}$. Thus if $\{\theta, \theta^1\}$ gives the CR structure of M and $\{\tilde{\theta}, \tilde{\theta}^1\}$ gives the CR structure of $\tilde{M}$, then

$$\begin{aligned} \phi^*(\tilde{\theta}) &= \lambda\theta \\ \phi^*(\tilde{\theta}^1) &= U\theta^1 + V\theta^{\bar{1}} + v\theta \end{aligned} \tag{4.16}$$

where λ is a non-zero real function and U, V, v are complex functions. For the proof of the theorem it suffices to show that $U|_p = 0$ or $V|_p = 0$ for $p \in M$. For simplicity we let $\tilde{\theta}$ also denote $\phi^*\tilde{\theta}$ and $\tilde{\theta}^1$ also denote $\phi^*\tilde{\theta}^1$. Then $\{\theta, \theta^1\}$ can be considered as defining CR structure on the same piece of $\mathbb{R}^3$ coordinatized by (x_1, x_2, x_3). Let $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a chain with respect to the $\{\theta, \theta^1\}$ structure. Thus $\gamma(t)$ satisfies the equations

$$\begin{aligned} &\theta^1(\tfrac{d}{dt}\big|_\gamma) = -\mu\theta\left(\tfrac{d}{dt}\big|_\gamma\right) \\ &\tfrac{d\mu}{dt}\big|_\gamma = (i\mu|\mu|^2 + B(\gamma(t), \mu(t)))\theta\left(\tfrac{d}{dt}\big|_\gamma\right) \end{aligned} \tag{4.17}$$

and the same for θ, θ^1, μ, B replaced by $\tilde{\theta}$, $\tilde{\theta}^1$, $\tilde{\mu}$, $\tilde{B}$ and $|B| < C(1 + |\mu|^2)$. Since the characteristic tangent spaces with respect to both CR structures coincide, a chain with $|\mu|$ large also has $|\tilde{\mu}|$ large. From now on we restrict attention to chains that satisfy both

$$|\mu| > 1 \quad \text{and} \quad |\tilde{\mu}| > 1$$

Along a chain one has

$$\theta^1 = -\mu\theta \quad \text{and} \quad \tilde{\theta}^1 = -\tilde{\mu}\theta$$

From this and (4.16) one has

$$-\mu(t)U(\gamma(t)) - \bar{\mu}(t)V(\gamma(t)) + v(\gamma(t)) = -\tilde{\mu}(t)\lambda(\gamma(t)) \tag{4.18}$$

Since λ is non-zero, there exist a constant C_1 independent of the particular chain such that

$$|\tilde{\mu}| < C_1|\mu|$$

provided $|\mu| \geq 1$. The same argument applied to ϕ^{-1} gives $|\mu| < C_2|\tilde{\mu}|$ provided $|\tilde{\mu}| \geq 1$.

Assume each chain γ is parametrized so that $|\frac{d\gamma}{dt}| = 1$ at p, where the norm is with respect to any fixed Riemannian metric. Then there exist some constant a and c such that at p

$$\frac{c}{|\mu|} < |\theta(\frac{d\gamma}{at})| < \frac{a}{|\mu|} \leq a$$

By differentiating (4.18) and using (4.17), one has, at p

$$\begin{aligned} & -(i\mu|\mu|^2 + B(\gamma,\mu))\theta\left(\frac{d}{dt}|_\gamma\right)U - (-i\bar{\mu}|\mu|^2 + \bar{B}(\gamma,\mu))\theta\left(\frac{d}{dt}|_\gamma\right)V \\ = \ & (i\tilde{\mu}|\tilde{\mu}|^2)\theta\left(\frac{d}{dt}|_\gamma\right)\lambda + \mathcal{O}|\mu| \end{aligned}$$

Here $\mathcal{O}|\mu|$ denotes a term bounded by $C|\mu|$ where C is independent of the chain γ and one has used that $\mathcal{O}|\tilde{\mu}|$ can be replaced by $\mathcal{O}|\mu|$. Further since $B = \mathcal{O}|\mu|^2$ and $\omega(\frac{d}{dt}|_\gamma) = \mathcal{O}(\frac{1}{|\mu|})$, one has

$$-i\mu|\mu|^2U + i\bar{\mu}|\mu|^2V = -i\tilde{\mu}|\tilde{\mu}|^2\lambda + \mathcal{O}|\mu|^2 \tag{4.19}$$

Substitute (4.18) into (4.19) to obtain, at p,

$$-i\mu|\mu|^2U + i\bar{\mu}|\mu|^2V = -\frac{i}{|\lambda|^2}(\mu U + \bar{\mu}V)(|\mu|^2|U|^2 + \mu^2U\bar{V} + \bar{\mu}^2\bar{U}V + |\mu|^2|V|^2) + \mathcal{O}|\mu|^2$$

which is an identity in μ. Comparing the coefficient of μ^3 one has

$$U^2\bar{V}|_p = 0$$

so either $U|_p$ or $V|_p$ is zero. $\square$

4.3 Some pathological behaviour of chains

In this section we mention some differences between the chains in CR geometry and the geodesics in Riemannian geometry. First it is useful to have the chain equations written directly in local coordinates. Starting with a hypersurface in normal form

$$\begin{aligned} v &= |z|^2 + \varphi(u)z^2\bar{z}^4 + \bar{\varphi}(u)\bar{z}^2 z^4 + \mathcal{O}|z|^7 \\ &= |z|^2 + F(z, \bar{z}, u) \end{aligned}$$

By (4.13) and (4.14) one has

$$\begin{aligned} b &= F_{zz\bar{z}} + \mathcal{O}|z|^5 = 8\varphi(u)\bar{z}^3 + 24\bar{\varphi}(u)z^2\bar{z} + \mathcal{O}|z|^4 \\ c &= F_{zz\bar{z}\bar{z}} + \mathcal{O}|z|^4 = 24(\varphi(u)\bar{z}^2 + \bar{\varphi}(u)z^2) + \mathcal{O}|z|^3 \\ l &= F_{zzz\bar{z}\bar{z}} + \mathcal{O}|z|^3 = 48\bar{\varphi}(u)z + \mathcal{O}|z|^2 \end{aligned}$$

The chain equation can be written as

$$\begin{aligned} dz &= -\mu\theta \\ d\mu &= (i\mu|\mu|^2 - (8\bar{\varphi}z^3 + 24\varphi\bar{z}^2 z + \mathcal{O}|z|^4)|\mu|^2 + 12i(\varphi(u)\bar{z}^2 + \bar{\varphi}(u)z^2 + \mathcal{O}|z|^3)\mu \\ &\quad -(8\varphi\bar{z} + \mathcal{O}|z|^2))\theta \end{aligned}$$

In local coordinates, by using (4.8),

$$\theta = (\frac{1}{2} + \mathcal{O}|z|^{12})du + (-\frac{i}{2}z + \mathcal{O}|z|^5)dz + (\frac{i}{2}z + \mathcal{O}|z|^5)d\bar{z}$$

Introducing the time parameter along a given chain for which

$$\theta(\frac{d}{dt}) = 1$$

Then

$$\begin{aligned} \frac{dz}{dt} &= -\mu \\ \frac{d\mu}{dt} &= i\mu|\mu|^2 - \bar{b}|\mu|^2 + \frac{1}{2}ic\mu - \frac{1}{6}\bar{l} \end{aligned}$$

and

$$(\frac{1}{2}+\mathcal{O}|z|^{12})\frac{du}{dt}+(-\frac{i}{2}\bar{z}+\mathcal{O}|z|^{5})\frac{dz}{dt}+(\frac{i}{2}z+\mathcal{O}|z|^{5})\frac{d\bar{z}}{dt}=1$$

These equations may be rewritten as

$$\begin{aligned}
\tfrac{du}{dt} &= 2 - i\bar{z}\mu + iz\bar{\mu} + A \\
\tfrac{dz}{dt} &= -\mu \\
\tfrac{d\mu}{dt} &= i\mu|\mu|^2 - (8\bar{\varphi}z^3 + 24\varphi\bar{z}^2 z)|\mu|^2 + 24i(Re(\varphi\bar{z}^2))\mu - 8\varphi\bar{z} + B
\end{aligned} \tag{4.20}$$

with $|A| \le C|\mu|\mathcal{O}|z|^5$ and $B \le C(|z|^2 + |z|^3|\mu| + |z|^4|\mu|^2)$. To study chains along which $|\mu| \to \infty$, one introduces another time parametrization. Let s and t be related by

$$\frac{dt}{ds} = \frac{1}{|\mu(t)|^2}$$

along a given chain which always has μ different from zero. Then (4.20) becomes

$$\begin{aligned}
\tfrac{du}{ds} &= \tfrac{2}{|\mu|^2} + \tfrac{iz}{\mu} - \tfrac{i\bar{z}}{\bar{\mu}} + a \\
\tfrac{dz}{ds} &= -\tfrac{1}{\bar{\mu}} \\
\tfrac{d\mu}{ds} &= i\mu - 8(\bar{\varphi}z^3 + 3\varphi|z|^2\bar{z}) + 24iRe(\varphi\bar{z}^2)(\tfrac{1}{\bar{\mu}}) - 8\varphi z(\tfrac{1}{|\mu|^2}) + b
\end{aligned} \tag{4.21}$$

where $|a| = \frac{|A|}{|\mu|^2}$ and $|b| = \frac{|B|}{|\mu|^2}$.

Chains in CR geometry differ from geodesics in Riemannian geometry in at least two ways. First, it is not true that any two arbitrary points can always be connected by a chain. Consider the following example from [BS].

Let (z,u) be the local coordinates on Q. The map $(z,u) \mapsto (rz, r^2u)$, $r \in \mathbb{R}$, is a CR diffeomorphism for any fixed $r \neq 0$. When $r \neq 1$, this map generates an infinite cyclic group G of diffeomorphisms. Let $\mathcal{D} = (Q - \{0\})/G$ denote the quotient under the equivalence relation $(z,u) \sim (r^n z, r^{2n}u)$, $n \in \mathbb{Z}$. Then $\mathcal{D}$ is a smooth manifold and has a CR structure locally the same as that of Q. In particular any chain of Q maps to a chain of $\mathcal{D}$ and all chains of $\mathcal{D}$ are so obtained. Let $\pi : Q - \{0\} \mapsto \mathcal{D}$ be the projection. Then $\pi(z,u) = [z,u] = \{(\zeta,\eta) | (\zeta,\eta) = (r^n z, r^{2n}u)$ for some $n \in \mathbb{Z}\}$.

Proposition 4.3.1 *There is no chain connecting [0,1] and [0, -1] in $\mathcal{D}$.*

Proof: Suppose such a chain γ existed. There would be a chain Γ in Q connecting $(0,1)$ and $(0,-r^{2n})$ for some n and for which $\Gamma - \{0\}$ projects to γ. But the only chain in Q between these two points in $\Gamma = \{(0,u)|u \in \mathbb{R}\}$ and $\pi(\Gamma - \{0\})$ consists of two disjoint closed curves in $\mathcal{D}$ and so cannot be γ. □

The second difference was discovered by Fefferman [Fef]. He has shown that the hypersurface

$$v = |z|^2 + u|z|^8$$

has a family of chains that "spiral" into the origin. We will not present his proof here, instead we make some remarks regarding limit points of chains. Let γ be a chain with some parametrization t, $-\infty < t < \infty$ and let p be a limit point in the sense that

(i) $\lim_{t\to\infty} \gamma(t) = p$

(ii) there does not exist a chain $\tilde{\gamma}$ containing p as an interior point and also containing the chain γ.

We show that $\gamma(t)$ must become more "horizontal" as it approaches p.

Proposition 4.3.2 $\lim_{t\to\infty} |\mu(t)| = \infty$

Proof: For each point q and each initial value μ_o, there is some $\varepsilon(q,\mu_o)$ for which the unique chain through q and having "slope" μ_o at q exits $B(\varepsilon, q)$. The function $\varepsilon(q,\mu_o)$ can be chosen to be continuous by results in ODE. Thus if q is restricted to a compact neighborhood of p and μ_o is restricted so that $|\mu_o| \leq C$ for some constant C, then the chain determined by q and μ_o exits some ball $B(\varepsilon, q)$ where ε is independent of q and μ_o.

Now assume on the contrary that there is a sequence $t_j \to \infty$ and a constant C such that $|\mu(t_j)| \leq C$. Choose j^* for which $dist(p,\gamma(t)) < \frac{\varepsilon}{2}$ for all t, $t > t_{j^*}$, where ε is as above. However the chain through $q = \gamma(t_{j^*})$ with slope $\mu_o = \mu(t_{j^*})$ must exit $B(\varepsilon, q)$. This leads to a contradiction. $\square$

Next we show that γ must have infinite arc length near a limit point.

Proposition 4.3.3 *Let γ be a chain on a 3 dimensional CR manifold $M \subset \mathbb{C}^2$ where $\mathbb{C}^2$ has the usual Euclidean structure of $\mathbb{R}^4$. If γ has a limit point then the arc length of γ is infinite.*

Proof: Starting with (4.21) and taking p to be the origin, then one has

$$\begin{aligned} \tfrac{dz}{ds} &= -\tfrac{1}{\bar{\mu}} \\ \tfrac{d\mu}{ds} &= i\mu + g(s) \end{aligned} \tag{4.22}$$

where $g(s)$ approaches zero as the point on the chain approaches the origin. Solving the second equation of (4.22) one has

$$\mu(s) = C_1 e^{is} + e^{is} \int_{s_o}^{s} e^{-i\sigma} g(\sigma) d\sigma$$

Hence

$$|\mu| < C_1 + C_2 s$$

where C_1 and C_2 are real constants and

$$\int_{s_o}^{\infty} \frac{1}{|\mu(s)|} ds = \infty$$

Consider the curve in the z-plane defined by the first equation of (4.22). This curve is a projection of the chain and so it suffices to show that this curve has infinite arc length. This is so since

$$\int_{s_o}^{\infty} |\frac{dz}{ds}| ds = \int_{s_o}^{\infty} |\frac{1}{\mu(s)}| ds = \infty$$

$\square$

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